FOUNDATIONS
OF ENGINEERING MECHANICS

A. Kovaleva

Optimal Control of Mechanical Oscillations

Springer

Foundations of Engineering Mechanics

A. Kovaleva, Optimal Control of Mechanical Oscillations

Springer-Verlag Berlin Heidelberg GmbH

Springer-Verlag Berlin Heidelberg GmbH

A. Kovaleva

Optimal Control of Mechanical Oscillations

Translated by V. Silberschmidt

With 20 Figures

Springer

Series Editors:
Prof. Dr. V. I. Babitsky
Loughborough University
Department of Mechanical Engineering
LE11 3TU Loughborough, Leicestershire
Great Britain

Prof. Dr. J. Wittenburg
Universität Karlsruhe (TH)
Institut für Mechanik
Kaiserstraße 12
D-76128 Karlsruhe / Germany

Author:
Prof. Dr. Agnessa Kovaleva
Russian Academy of Sciences
Mechanical Engineering Research Institute
101830 Moscow / Russia
e-mail: agnessa@meri.msk.ru

Translator:
Dr. Vadim Silberschmidt
Technische Universität München
Lehrstuhl A für Mechanik
Boltzmannstraße 15
85747 Garching / Germany

Library of Congress Cataloging-in-Publication Data
Kovaleva, A.S. (Agnessa Solomonova), Optimal control of mechanical oscillations
A. Kovaleva; translated by V. Silberschmidt (Foundations of engineering mechanics)
Includes bibliographical references and index
ISBN 978-3-642-08476-8 ISBN 978-3-540-49258-0 (eBook)
DOI 10.1007/978-3-540-49258-0

1. Oscillations. 2. Control theory. I. Title. II. Series
QA865.K69 1999
531'.32--dc21 99-17773

ISBN 978-3-642-08476-8

Typesetting: Camera-ready copy from translator
Cover-Design: de'blik, Berlin
SPIN 10694259 62/3020 5 4 3 2 1 0 Printed on acid-free paper

Contents

Introduction

The optimal control theory suggests general methods for a design of systems according to a given quality criterion. Despite of the existence of general approaches, a complexity of real systems still demands an elaboration of special methods with account for specific features of a concrete type of problems.

Problems of the oscillatory systems' control form a special class. The suggested book deals with the two aspects: an initiation of optimal oscillatory (and, in particular, periodic) regimes and a study of possibilities of oscillatory systems' control for random Disturbances.

Problems of the oscillation control were analyzed in the books of Tchernousko, Akulenko and Sokolov [134], Plotnikov [111], Troitsky [125] and in numerous articles (see a detailed list of references in [135]). The main investigation method for deterministic oscillatory systems was the Pontryagin maximum principle in combination with asymptotic methods of the oscillation theory. However, an utilization of the maximum principle is connected with considerable difficulties linked with the high order of such systems. Thus, a control problem for the system with n degrees of freedom needs a solution of the boundary–value problem with $4n$ equations. In a general case the problem is complicated by the presence of nonlinear links, and in the analysis of vibroimpact systems – by the conditions of speed discontinuity.

At the same time, in real control problems the main constraints and functional depend, as a rule, on the movement of one or of some characteristic points of the system and do not contain all the generalized coordinates. For instance, in a control of a manipulator it is necessary to realize an optimal trajectory of an actuator, and constraints for a drive are not introduced. In a vibroimpact protection design, displacements and accelerations of one or some points of the object are minimized or limited, while the structure of the object can be determined only experimentally (by dynamic compliance for some frequencies). and not be described by ordinary differential equations.

In this case another method – based on the description by integral equations of periodic regime – is more suitable for the analysis of the periodic movement. A reduction of the problem of the periodic analysis to a study of integral equations of Hammerstein–type was suggested by E.N. Rosenwasser [115,116]. In this approach one or several equations for coordinates under study can be separated, and the control problem can be reduced to the minimization problem for a functional with constraints in the integral–equation form. Still, a number of equations is determined not by the degree of freedom but by problem's constraints.

Chapter 1 treats the basic principles of the integral equation method, to the formulation of the maximum principle and the solution of several problems of the periodic movement control. This method is found to be effective also for a solution of the problem of an optimal high–speed action, where the movement is considered at a finite time interval and does not possess a repetition property. Each process determined for a finite time interval is expandable into Fourier series for this interval. So, each movement can be described by an integral equation of the periodic regime for the interval under study, and the problem of the high–speed action can be reduced to the one of the periodic control.

Chapter 2 consider an application of these results to control problems of vibro-impact systems. A description of periodic regimes of such systems by means of integral equations was suggested by M.Z. Kolovsky and V.I. Babitsky [16,23]. The integral equation method was found to be especially effective in those control problems, where the use of traditional schemes was complicated by the necessity of an account for conditions of an impact and speed discontinuity. Thus, the control problems for systems with one degree of freedom at impact were mainly studied [16, 67]. An optimization procedure based on integral equations is not linked with discontinuity conditions and does not depend on the dimension of a linear part of the system, but is determined by a control number and the character of problem's constraints.

In Chapter 3 the control problems of systems with weak control are considered. Here, sufficient effects can be achieved by means of the weak control excitation. The features of the solution of high–speed–action problems in such systems are shown, the optimal periodic regimes are obtained.

The main theorems of the maximum principle and of the averaging method used in Chapter 3 are given in Appendix.

Chapters 4 and 5 deal with the study of oscillatory systems with random disturbances.

In extensive literature on the control of stochastic systems, it is usually considered that the disturbance is either a white noise or a result of its transition through the filter, and that the system thus can be described by means of Itô's stochastic differential equations. Such considerations allow the formulation of dynamic programming equations and the suggestion of the control method. Still, an assumption about a Markov character of the excitation requires an unjustified detailing of its description for the solution of applied problems and causes a considerable complication of the problem. Even for the most simple case of the excitation in the form of a stationary process with the rational fractional spectral density, the motion equations should be supplemented with the filter equations. In such a case the system's dimension grows and the computational difficulty linked with a solution of the problem of dynamic programming sharply increases. At the same time, it was shown in the works of R.L. Stratonovic [120, 121], R.Z. Has'minskii [131, 132], which where later developed by H.J. Kushner [179] and others, that for definite assumptions on the disturbance character – not necessary of a Markov type – a solution of the system with disturbances is approximated by a diffusion process, i.e. by a solution of the averaged system of Itô's equations.

In Chapter 4 the basic principles of the diffusion approximation method are given, while Chapter 5 deals with its application to control problems. The main idea is a replacement of the initially perturbed system by a limit diffusion one and the formulation of an optimal control for trajectories of the limit system. Quasi-optimality of the found control with respect to the initial system is proved.

The main material of the book is based on some assertions of the optimal control theory and the disturbance theory. The information necessary for understanding is given in Appendix.

All the theoretical propositions of the book are illustrated by examples with exact mechanical context. Let here also note, that the main content of the book is mathematical foundations, applications of which are not limited only to given model examples.

Some of results of this book were discussed by the author with V.I. Babitsky, V.Sh. Burd, V.F. Zhuravlev, A.A. Pervozvanskii. The author is very grateful to all of them for their advises and critical comments. The author is especially indebted to her teacher M.Z. Kolovsky, without advises and support of whom this book would not be written.

1 Optimal Periodic Control. The Integral Equations Method

The quality of mechanism's functions is usually characterized by demands to the motion of some of its points. Main restrictions which should be accounted in design of guided mechanisms and vibroimpact systems are formulated in [36,83].

Thus, a programmed position control realized in various transporting systems including robot-manipulators should ensure a displacement of an actuator to a given point in a fixed (or minimal) time. In other words, a motion law for all elements of a system with several degrees of freedom should be determined by the actuator motion. The design aim of optimal vibroimpact systems is a minimization of displacements or accelerations of some characteristic points of the object (device, mechanisms, etc.), the structure of which could be characterized only by experimental data (dynamic compliance at definite frequencies).

Thus, the quality of guided systems is characterized by the motion of several points, a number of which is usually less than a degree of freedom of the system. In traditional formulations of periodic control problems, the system dynamics is described by differential equations, while periodicity conditions are treated as additional relations linking generalized coordinates and velocities at the begin and end of the period. The maximum principle which was directly formulated for periodic problems in [160] serves as a necessary condition. Such formulation does not differs, as a matter of fact, from the traditional problem of functional minimization, and demands the solution of a system of equations of the maximum principle, in which boundary conditions have the form of periodicity conditions. Here, for a system with n degrees of freedom, the system of $4n$ equations should be solved, no matter what generalized coordinates are in the functional and constraints of the problem.

At the same time, another method of description of periodic movements is possible which is not linked with traditional differential–eqation form and which allows us to separate motion equations for one or several generalized coordinates. If a system contains a linear part (this condition is usually always fulfilled for mechanical systems), then its motion can be described by integral equations of periodic movement [115,116]. The kernels of these equations are determined by the linear part of the system. This method is especially effective in problems of the optimal control, if constraints and the functional of the problem depend on a trajectory of one characteristic point or of actuator. Here, one integral equation of periodic movement can be often singled out for a certain coordinate, independent of the system's degree of freedom, and the optimal control problem can be reduced to minimization of respective functional with constraints in form of integral equations.

This Chapter is dedicated to the method of integral equations for solution of optimal periodic control problems.

In Section 1.1 necessary information on dynamic characteristics of oscillatory systems is given.

In Section 1.2 general ideas of the integral equations method are presented, integral equations for periodic motions are formulated, and the main properties of kernels are discussed.

The subsequent material is directly connected with optimal control problems. In Section 1.3 necessary conditions of optimality for periodic regimes are formulated; different types of constraints for trajectories and control in periodic control problems are analyzed.

Examples of problem solution for periodic control are given in Sections 1.4 and 1.5. In Section 1.4 linear stationary systems are studied. The detailed analysis of the linear system for a quadratic criterion is carried out. The analytic solution for this problem is given in the form of the programmed control.

In Section 1.5 problems of an optimal high–speed action for linear problems are analyzed. The method of reduction of such problems to periodic control problems for movements with non-stationary period is proposed, conditions for determination of the optimal oscillation period are given. Problems of organization of the cyclic movement, characteristic to manipulators transporting loads in a conveyor system are of main interest. In such a case the system moves in the load direction and returns into initial state without load, i.e., with different dynamic characteristics. The way for design of the optimal high–speed control for each part of the cycle is proposed, the main relations for a cycle duration are obtained.

In Section 1.6 optimal periodic control problems for non-linear and non-stationary systems are studied. The systems are considered to be close to linear stationary ones. The scheme of successive approximations is given and the error estimation for the determination of trajectories and control parameters for this scheme are discussed.

1.1 Linear Systems, Basic Definitions

1.1.1 General Concepts and Definitions

For a study of mechanical systems of complicated structure it is suitable to use the concepts of the control theory. A deviation from the traditional treating of the oscillation theory as a part of the theory of differential equations permits a deeper understanding of the main properties of mechanical control systems, including also the systems described by differential equations. Besides, the use of transfer functions makes calculations more compact, and respective results can be more easily interpreted. A consequent treatment of the theory of mechanical oscillations in terms of transfer functions is given in monograph [83]; a series of subsequent studies on dynamics of machines and a theory of active vibroprotective systems [16, 36, 128] approve the effectiveness of such an approach.

Each linear system is an idealization of a real object. This type of systems was thoroughly studied thanks to a wide spectrum of their applications. Let us mention some properties of linear continuous systems [60].

Dynamics of the system is described as

$$\begin{aligned} &L(p,t)y = M(p,t)u\,, \\ &L(p,t) = a_n(t)p^n + \ldots + a_0(t)\,, \\ &M(p,t) = b_m(t)p^m + \ldots + b_0(t)\,, \end{aligned} \tag{1.1}$$

where $p = d/dt$ is a differentiation operator. In a general case, y and u are vector values and coefficients a_j and b_j are matrices of respective order.

Let us analyze in details a particular case: a one-dimensional system, i.e., y and u are scalars and $n > m$. Then a system's response can be presented in the form of a sum of responses at a zero input $y_0(t)$ and a zero initial state $y_*(t)$: $y_*(t) = y_0(t) + y_*(t)$.

Here $y_0(t)$ is a solution of a homogeneous equation

$$L(p,t)y = 0$$

with a fixed vector of initial conditions $s(t_0) = \{y(t_0), \ldots, y^{(n-1)}(t_0)\}$; a function $y_0(t)$ can be presented in a form

$$y_0(t) = \sum_{j=1}^{n} h_j(t,t_0)y^{(j-1)}(t_0) = (H(t,t_0), s(t_0)),$$

where $\{H(t,t_0) = h_1(t,t_0), \ldots, h_n(t,t_0)\}$ is a vector of the basis function. Functions $h_j(t,t_0)$ form a linearly independent system of solutions of the homogeneous equation

$$L(p,t)h_j = 0 \tag{1.2}$$

with initial conditions: for $t = t_0$

$$\frac{d^i h_j}{dt^i} = 0, \quad i = 0,1,\ldots,n-1\,; \quad i \neq j-1\,, \quad \frac{d^{j-1}h_i}{dt^{j-1}} = 1\,.$$

The function $h_n(t)$ satisfying initial conditions: for $t = t_0$

$$\frac{d^i h_n}{dt^i} = 0, \quad i = 0,1,\ldots,n-2\,; \quad \frac{d^{n-1}h_n}{dt^{n-1}} = 1\,, \tag{1.3}$$

is called a *Cauchy function* of the system; the Cauchy function is interpreted as a response of a system

$$L(p,t)h_n = u$$

to an impulse excitation $u = \delta(t - t_{0+})$ for zero initial conditions (here $\delta(t)$ is a Dirac function [43,70,137]).

The solution of Eq. (1.1) at zero initial conditions has the form

$$y_* = \int_{t_0}^{t} h(t,s)u(s)ds. \tag{1.4}$$

The kernel $h(t,s)$ is called an *impulse transition function of a closed system* and can be determined as a response of the system (1.1) to an impulse excitation of the kind $u = \delta(t - t_{0+})$. If $M(p,t) = 1$, then $h(t,s) = h_n(t,s)$.

If coefficients of Eq. (1.1) do not depend on t then basis functions $h_j(t,t_0)$ and the impulse transition function $h_j(t,t_0)$ of the system depends only on the difference of the variables

$$h_j(t,t_0) = h_j(t - t_0), \qquad j = 1,\dots,n,$$
$$h(t,t_0) = h(t - t_0),$$

and the solution of the system (1.1) can be written in the form

$$y(t) = \sum_{j=1}^{n} h_j(t - t_0)y^{(j-1)}(t_0) + \int_{t_0}^{t} h(t - s)u(s)ds.$$

1.1.2 Transfer Function of Linear System. Stable and Physically Realizable Systems

A concept of physical realizability is very important for practical applications. A system is called *physically realizable* when its response (1.4) at the moment t depends only on current (for $s = t$) and past ($s < t$) input values and does not depend on its future ones $s > t$. This means for a stationary system that the system is physically realizable only then and only then, when the impulse transition function $h(t)$ turns into zero for $t < 0$: $h(t) = 0$ for $t < 0$. A condition of physical realizability of a non-stationary system is reduced to a demand $h(t,s) = 0$ for $s > t$.

Consider $h(t)$ to be the impulse transition function of the stationary physically realizable system. Then a Laplace transform

$$H(p) = \int_0^{\infty} h(t) e^{-pt} dt, \tag{1.5}$$

or, in other form,

$$H(p) = \int_{-\infty}^{\infty} h(t-s) e^{-p(t-s)} ds$$

is called a *transfer function*; for the physically realizable system $h(t-s) = 0$ for $t < s$.

In a general case of a non-stationary system with the impulse transitive function $h(t,s)$ the transfer function depends on time:

$$H(p,t) = \int_{-\infty}^{\infty} h(t-s) e^{-p(t-s)} ds; \tag{1.6}$$

for physically realizable system $h(t,s) = 0$ for $t < s$.

For a system

$$L(p)y = M(p)u \tag{1.7}$$

a transfer function has a form

$$H(p) = \frac{M(p)}{L(p)}, \tag{1.8}$$

and an impulse response can be found as the inverse Laplace transform (1.5).

Eq. (1.7) can by written with the help of the transfer function as

$$y = H(p)u, \tag{1.9}$$

and the system can be schematically presented in the following form (Fig. 1.1). Eq. (1.2) describes the system's response for a zero initial state. For a stationary system this response is presented by a convolution

$$y_* = \int_{t_0}^{t} h(t-s) u(s) ds.$$

The transfer function connects a Laplace transform of input and output

$$Y_*(p) = H(p) U(p),$$

where $Y_*(p)$ and $U(p)$ are Laplace transforms of the functions $y_*(t)$ and $u(t)$.

A distribution of poles of the transfer function allows us to estimate the system's stability. Now recall the main definitions.

A system described by the equation

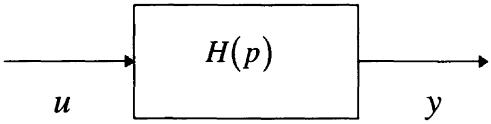


Fig. 1.1

$$L(p,t)y_0 = 0$$

is called stable, if a vector of its states $s(t) = \{y_0(t),\ldots, y_0^{n-1}(t)\}$ is bounded for an arbitrary initial state $s(t_0)$. If also

$$\lim_{t \to 0} s(t) = 0,$$

then the system is asymptotically stable. If a linear system is asymptotically stable, then all the solutions of the full system

$$L(p,t)y = M(p,t)u$$

are bounded when the excitation $f = Mu$ is also bounded.

For the asymptotic stability of the system (1.7) it is necessary and sufficient that all the solutions of the characteristic equation $L(p) = 0$ are in a left half-plane. Various criteria of the asymptotic stability are given in a vast literature on the automatic control.

1.1.3 Steady and Periodic Solutions of a Linear System

Let us introduce a concept of a steady state and of a steady response for a linear system [60].

A steady state of the system (at a zero input) is a limit ($t \to \infty$) state γ, which is reached by the system at the zero input and which does not depend on the initial state $s(t_0)$. For the systems under study, the limit state

$$\gamma = \lim_{t \to \infty} s(t), \quad s(t) = \{y_0(t),\ldots, y_0^{n-1}(t)\},$$

where $y_0(t)$ is a solution of the homogeneous equation

$$L(p,t)y = 0$$

for the initial state $s(t_0)$. It is evident that for asymptotically stable systems there exists the unique steady state $\gamma = 0$. If the system is not stable or is non-asymptotically stable, then it is impossible to determine the limit which is independent of initial conditions.

The steady response of the system is determined in the following way [60]. It is considered that the initial state $s(t_0)$ coincides with the steady state γ and the initial moment $t_0 \to -\infty$. The response to input, determined as described above, is called the *steady response*. The physical meaning of such a definition is following: all transition processes which start at $t_0 \to -\infty$ fade and the system's motion "steadies".

It is evident, that the concept of the steady response has sense for a linear system only in such a case, when the system is asymptotically stable and $\gamma = 0$. Owing to $\gamma = 0$, there exists a steady response

$$\bar{y}(t) = \int_{-\infty}^{t} h(t,s)u(s)ds.$$

If the system is stationary, $h(t,s) = h(t-s)$, then its response can be transformed to

$$\bar{y}(t) = \int_{0}^{\infty} h(s)u(t-s)ds.$$

A transfer function of the asymptotically stable system can be determined by means of a steady response of the system to the excitation e^{pt}. Indeed, considering $u(t) = e^{pt}$ in Eq. (1.6), we get

$$\bar{y}(t) = e^{pt}\int_{0}^{\infty} h(s)e^{-ps}ds = e^{pt}H(p).$$

It follows, in particular, that the steady response of the system to the excitation $e^{i\omega t}$

$$\bar{y}(t) = e^{i\omega t}H(i\omega) \tag{1.10}$$

is also a periodical function with a period of $T = 2\pi/\omega$. A complex function $H(i\omega)$ is called a *frequency characteristic* of the system.

If a stationary system is not asymptotically stable, then Eq. (1.10) still keeps its meaning: if $u(t) = e^{i\omega t}$ and $L(i\omega) \neq 0$, then there exists a periodical solution of the system (1.4): $y_T = e^{i\omega t}H(i\omega)$. This can be easily approved by the substitution of $u(t) = e^{i\omega t}$ and $y_T = H(i\omega)e^{i\omega t}$ in Eq. (1.4). The periodical solution should not be considered as a special case: it is a partial solution of the non-homogeneous equation which satisfies initial conditions $y^{(j-1)}(0) = (i\omega)^{j-1} H(i\omega), j = 1,...,n.$

If the excitation has a form of the sum of harmonics

$$u(t) = \sum_{k=1}^{N} u_k e^{i\omega_k t} ,$$

and $L(i\omega_k) \neq 0$, then there exists a polyharmonic solution

$$y(t) = \sum_{k=1}^{N} u_k H(i\omega_k) e^{i\omega_k t} , \tag{1.11}$$

which contains the same harmonics. This expression can be considered as a partial solution of the non-homogeneous equation for definite initial conditions. The steady solution for the asymptotically stable system coincides with the obtained polyharmonic one.

1.1.4
Dynamic Characteristics of Mechanical System

The concepts of input, output and transfer function have a physical meaning for mechanical systems. Consider an arbitrary mechanical system (Fig. 2) with applied forces $F_k(t)$; the projections of these forces to the coordinate axis form a $3k$-dimensional vector $F(t)$. Let $\xi_1(t),...,\xi_s(t)$ be displacements of system's points, caused by the applied excitation.

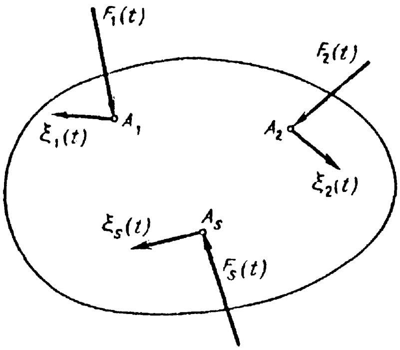


Fig. 1. 2

The number of independent co-ordinates $\xi_i(t)$ which determine the system's position in space is called a *degree of freedom* and the vector $\xi(t)$ is called a *state of the system.*

If the initial sate of the system $\xi(t_0)$ and equations linking the applied forces and displacements are known, then it is always possible to determine the state of the object at the moment t.

For formulation of the equation, a dynamic model of the system should be chosen (i.e., objects should be presented as a set of inertial, elastic, and damping elements) and a structure of control system should be known. Such a subdivision into elements can not be always performed. Besides, the model choice should depend on the excitation: the wider is the excitation band and the higher are the frequencies that it contains, the larger number of degrees of freedom should have the model system. Furthermore, not always all the displacements linked to all degrees of freedom are of interest: The system's behavior is estimated, as a rule, by the movement of some of its points. For simplification of the description, a concept of *operators of dynamic stiffness* and *dynamic compliance* [83] linking forces and displacements of various points are introduced.

Suppose that linear equations of the system's movement can be formulated as a dependence of the state of the system $\xi = \{\xi_1, ..., \xi_n\}$ on applied forces $F = \{F_1, ..., F_m\}$, and that these equations can be transformed to

$$\xi = L(p)F \,, \quad \xi_j = \sum_{k=1}^{m} l_{jk}(p)F_k \,, \quad j = 1,...,n.$$

The operator $L(p)$ is a *matrix transfer function* of the system; $l_{jk}(p)$ are the elements of the matrix $L(p)$.

Let all the forces but one – $F_r(t)$ – be equal zero. Then the relation between a force applied at the point A_r and a displacement of the point A_j

$$\xi_j = l_{jr}(p)F_r \tag{1.12}$$

is expressed by an operator $l_{jr}(p)$. The operator $l_{jr}(p)$ is called the *operator of dynamic compliance*, and the inverse operator

$$d_{jr}(p) = l_{jr}^{-1}(p) \tag{1.13}$$

is called an *operator of dynamic stiffness* of the system [83].

If $p = i\omega$ then the frequency characteristics $d_{jr}(i\omega)$, $l_{jr}(i\omega)$ are called *dynamic stiffness* and *dynamic compliance* at this frequency. These characteristics can be obtained experimentally.

Let the force $F(t)$ be applied at some point *A*, and consider that $v(t)$ is a projection of this point's displacement to the force direction. Characteristics $d_A(i\omega)$, $l_A(i\omega)$ linking the force *F* and displacement *v* are called the dynamic compliance and dynamic stiffness at point *A* [83].

An analysis of a transition process in the system is reduced to a solution of Eq. (1.14) with account for initial conditions, and this problem is in principle equivalent to the solution of the full system of state equations. At the same time, if the force $F_r(t)$ is changed according to a harmonic law

$$F_r(t) = F_{0r} \cos \omega t \,,$$

then for a steady response one can get (accounting for Eq. (1.11))

$$\xi_j(t) = \xi_{0j} \cos(\omega t + \varphi_j), \tag{1.14}$$

where $\xi_{0j} = |l_{lr}(i\omega)| F_{0r}$, $\varphi_j = \arg l_{jr}(i\omega)$.

Let us describe some general properties of dynamic characteristics [83]. The movement equations for a general case of a stationary linear mechanical system get the form

$$D(p)x = F(t) \ . \tag{1.15}$$

Here $D(p)$ is a matrix of dynamic stiffness with elements

$$d_{rq} = M_{rq}p^2 + n_{rq}p + c_{rq} ;$$

M_{rq}, n_{rq}, c_{rq} (r, q = 1,...,n) are coefficients of positively determined quadratic forms

$$T = \frac{1}{2}\sum_{r,q=1}^{n} M_{rq}\dot{x}_r\dot{x}_q \ , \qquad B = \frac{1}{2}\sum_{r,q=1}^{n} n_{rq}\dot{x}_r\dot{x}_q \ ,$$

$$\Pi = \frac{1}{2}\sum_{r,q=1}^{n} c_{rq}x_r x_q$$

characterizing the kinetic energy, dissipative function and potential energy of the system, respectively [42]. If dissipation in the system is small, then all the elements n_{rq} are small values.

The coordinate x_k can be expressed in terms of respective components of the matrix of dynamic compliance:

$$x_k = \sum_{j=1}^{n} l_{kj}(p)F_j \ . \tag{1.16}$$

Here

$$l_{kj}(p) = R_{kj}(p)\Delta^{-1}(p) , \tag{1.17}$$

where $R_{kj}(p)$ is an algebraic complement of the element $d_{kj}(p)$ of the dynamic compliance matrix $D(p)$, $\Delta(p) = \det D(p)$. The expressions $\Delta(p)$ and $R_{kj}(p)$ are polynomials of p of the order n and not higher than $2n$-2, respectively.

Let us re-write Eq. (1.15) in main coordinates. Then [42]

$$\ddot{z}_r + \sum_{q=1}^{n} b_{rq}\dot{z}_q + \Omega_r^2 z_r = u_r(t) , \tag{1.18}$$

where $z_r = \sum_{q=1}^{n} A_{rq}x_q$, $u_r = \sum_{q=1}^{n} A_{rq}F_q(t)$. Here Ω_r are the eigenfrequencies of a conservative system, A_{rq} are coefficients of the r-th form of oscillations, the values b_{rq} are small. The characteristic equation of the system then get the form

$$\Delta(p)=\begin{vmatrix} p^2+b_{11}p+\Omega_1^2 & \dots & \dots & b_{1n}p \\ b_{21}p & p^2+b_{22}p+\Omega_2^2 & \dots & b_{2n}p \\ \dots & \dots & \dots & \dots \\ b_{n1}p & b_{n2}p & \dots & p^2+b_{nn}p+\Omega_n^2 \end{vmatrix}=0. \quad (1.19)$$

By opening the determinant and neglecting the products of small coefficients, the expression for the roots of the characteristic equation can be obtained

$$p_j=-r_j\pm i\Omega_j, \quad r_j=b_{jj}/2>0. \quad (1.20)$$

If a harmonic excitation $F_j=F_0\cos\omega t$ is applied to the system, and $F_k(t)\equiv 0$ for $k\neq j$, then it follows from Eq. (1.16)

$$x_k(t)=\left|l_{kj}(i\omega)\right|F_0\cos(\omega t+\varphi_k),$$
$$\varphi_k=\arg l_{kj}(i\omega)=\arctan\left[\operatorname{Im} l_{kj}(i\omega)/\operatorname{Re} l_{kj}(i\omega)\right]. \quad (1.21)$$

It is clear that Ω_j are resonance frequencies of the system: for $\omega=\Omega_j$ a small multiplier $2r_j i\Omega_j$ appears in a denominator of (1.17), (1.21). It means, that a dynamic compliance modulus becomes large and the amplitude of oscillations sharply increases.

1.2
Periodic Green's Functions and Periodic Motions of Linear Systems

1.2.1
Periodic Regime of Linear Systems

Let $H(p)$ be a transfer function of the stationary system (1.7)

$$y=H(p)u, \quad H(p)=M(p)/L(p), \quad (1.22)$$

$u(t)$ is a periodic input signal, $u(t)=u(t+T)$.

Each periodic system can be confronted with its Fourier series [43, 137]

$$u(t)\propto\sum_{k=-\infty}^{\infty}u_k e^{ki\omega t}, \quad \omega=2\pi/T, \quad u_k=\frac{1}{T}\int_0^T u(t)e^{-ki\omega t}dt. \quad (1.23)$$

A Dirichlet theorem is true: If the function $u(t)$ is piecewise continuous on the interval $[0,T]$ and has only finite number of discontinuity points in this interval, then its Fourier series is converged to the sum $u(t)$ in each point of continuity and to the sum $[u(t+0)+u(t-0)]/2$ in each discontinuity point. Hence, if the function $u(t)$ is continuous, then the series converges, and Eq. (2.23) is the strict equality; if the function is not continuous, then the equality is possible only in points of continuity.

The definite meaning of diverged Fourier series can be considered in terms of the theory of generalized functions [137]. Let us study a periodic generalized function, important for following applications:

$$\delta^T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT), \tag{1.24}$$

which presents a periodic sequence of impulses (Fig. 1.3). A formal Fourier series

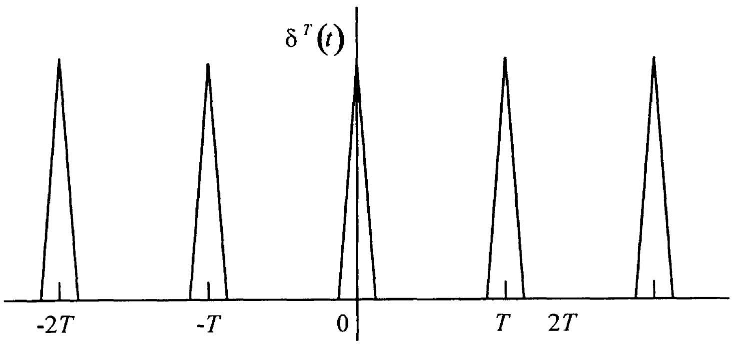


Fig. 1.3

of this function [137]

$$\delta^T(t)\propto\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{ki\omega t} \tag{1.25}$$

diverges, naturally, in each point. Still, all the properties and operations, which are determined for (1.24) remain in a case of the replacement of $\delta^T(t)$ by its Fourier series [147]. Below, an equality sign is used in a case of a function replacement by its Fourier series, understanding an equality in a generalized sense.

The following statement holds:

Theorem 1.1 [115]. *If there is no pole* $p=ki\omega$ $(\omega=2\pi/T)$ *among the roots of the function L(p), then the T-periodic response of the system* (1.22) *to the excitation* $u(t)=u(t+T)$ *can be described by the equation*

$$y_T(t)=\int_0^T\chi_1(t-s)u(s)ds, \tag{1.26}$$

where

$$\chi_1(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}H(ki\omega)e^{ki\omega t} \tag{1.27}$$

is the periodic Green's function of the first kind for the system (1.22).

It is sufficient for the formal proof to substitute (1.27) into (1.26) and to compare it with (1.11). The detailed proof of the Theorem 1.1 is given in [115].

Replacing the last integral by its Fourier series u_k, the desired equality can be obtained.

If the system is asymptotically stable, then the obtained solution coincides with a stationary response. According to (1.10), the stationary response of the system (1.22) to the periodic excitation (1.23) can be written in the form

$$\bar{y}(t)=\frac{1}{T}\int_0^{\infty}h(s)\sum_{k=-\infty}^{\infty}u_k e^{ki\omega(t-s)}ds\,.$$

If the series converges, then, replacing the order of summation and integration, we can get

$$\bar{y}(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}H(ki\omega)u_k e^{ki\omega t}=y_T(t).$$

1.2.2 Main Properties of the Periodic Green's Function

In principle, each periodic solution can be presented as a Fourier series. Still, it is more convenient to get the reasonable practical estimations, when a closed presentation of the function and a description of the periodic solution are reduced to calculation of the integral (1.26).

A detailed investigation of properties of the periodic Green's function is given in [115,116]. For the problems of the theory of oscillatory systems, only some of these properties, necessary for a solution of several equations, are essential.

Comparison of (1.25) and (1.27) shows, that the function $\chi_1(t)$ can be interpreted as a periodic response of the system (1.22) to the excitation $\delta^T(t)$.

For a system with a fraction-rational transfer function $H(p)=M(p)/L(p)$, the series (2.27) can be presented in a closed form [99]. Let $M(p)=b_m p^m+...+b_0$, $L(p)=p^n+a_{n-1}p^{n-1}+...+a_0$. If $m<n$ and all the roots of the function $L(p)$ are simple, then the function $H(p)$ can be expanded into simple fractions

$$H(p)=\sum_{r=1}^{n}\frac{M(p_r)}{L'(p_r)(p-p_r)},\qquad L'(p_r)=\left[\frac{dL}{dp}\right]_{p=p_r}, \tag{1.28}$$

i.e.,

$$\chi_1(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H(ki\omega)e^{ki\omega t} = \frac{1}{T}\sum_{r=1}^{n}\frac{M(p_r)}{L'(p_r)}\sum_{k=-\infty}^{\infty}\frac{e^{ki\omega t}}{ki\omega - p_r} \tag{1.29}$$

with a demand $p_r \neq ki\omega$.

The inner series can be calculated. As far as a Fourier series

$$g(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\frac{e^{p_r T}-1}{p_r - ki\omega}e^{ki\omega t} \tag{1.30}$$

corresponds to a periodic function $g(t)$ introduced for the interval $0 < t < T$ by the equation

$$g_T(t) = e^{p_r t}, \tag{1.31}$$

then, comparing expansions for $\chi_1(t)$ and $g(t)$, we note that the function $\chi_1(t)$ can be presented as

$$\chi_1(t) = \sum_{r=1}^{n}\frac{M(p_r)}{L'(p_r)}\frac{e^{p_r t}}{1-e^{p_r T}}, \qquad 0 < t < T \tag{1.32}$$

(the index T is omitted below).

The periodic regime determination can be reduced to the calculation of the integral (1.26), expressing $\chi_1(t)$ in the finite form (1.32). It is necessary to remember that the presentation of $\chi_1(t)$ in the form of (1.32) leads to periodic solution for the interval $0 < t < T$; for $t > T$ this solution gives a respective periodic extension.

If $p_1, \ldots, p_l$ are various roots of the equation $L(p) = 0$ and $\nu_1, \ldots, \nu_l$ are their multiplicities,

$$\sum_{s=1}^{l}\nu_s = r,$$

then the expansion of the function $H(p)$ into simple fractions has the form [99]

$$H(p) = \frac{M(p)}{L(p)} = \sum_{i=1}^{l}\sum_{k=1}^{\nu_i}\frac{c_{ik}}{(p-p_i)^k}, \tag{1.33}$$

where

$$c_{ik} = \frac{1}{(\nu_i - k)!}\frac{d^{\nu_i - k}}{dp^{\nu_i - k}}\left[\frac{M(p)(p-p_i)^{\nu_i}}{L(p)}\right]_{p=p_i}, \tag{1.34}$$

and the sum of the periodic Green's function for the interval $0 < t < T$ is the finite relation [99,115]

$$\chi_1 = \sum_{i=1}^{l} \sum_{k=1}^{\nu_i} \frac{c_{ik}}{(k-1)!} \left[\frac{d^{k-1}}{dp^{k-1}} \frac{e^{pt}}{1-e^{pT}} \right]_{p=p_i}, \quad 0 < t < T. \tag{1.35}$$

Let us demonstrate some general properties of the periodic Green's function, resulting from its presentation by the Fourier series.

If $m < n-1$, then the function $\chi_1(t)$ is continuous, if $m = n-1$, then $\chi_1(t)$ has discontinuities of the first kind in points $t = kT$. The proof of this proposition, common for all Fourier series, can be found in multiple books (see, for instance, [137]): It is expedient to give here only a relation for a discontinuity value $\Delta = \chi_1(T) - \chi_1(0)$ (strictly saying, it is necessary to write $\Delta = \chi_1(T_-) - \chi_1(0_+)$).

From (1.32) we have

$$\chi_1(0) - \chi_1(T) = \sum_{r=1}^{n} \frac{M(p_r)}{L'(p_r)}, \tag{1.36}$$

where

$$M(p) = b_m p^m + \ldots + b_0, \quad L(p) = p^n + a_{n-1} p^{n-1} + \ldots + a_0.$$

Such sums can be presented by coefficients of the polynomial $L(p)$ [99]

$$\sum_{r=1}^{n} \frac{p_r^s}{L'(p_r)} = \begin{cases} 1, s = n-1, \\ 0, 0 \le s \le n-2, \end{cases}$$

i.e.,

$$\chi_1(0) - \chi_1(T) = b_{n-1}. \tag{1.37}$$

Furthermore, the equation (1.37) holds also for a general case of even roots [115].

Let us formulate the form of $\chi_1(t)$ for a mechanical system with one degree of freedom:

$$\ddot{y} + 2b\dot{y} + \Omega^2 y = u, \tag{1.38}$$

i.e., $L(p) = p^2 + 2bp + \Omega^2$, $M(p) = 1$.

If $b^2 > \Omega^2$, then $p_{1,2} = -b \pm b_1$, $b_1 = \sqrt{b^2 - \Omega^2}$,

$$\chi_1(t) = \frac{e^{-bt}}{b_1} \frac{\operatorname{sh} b_1 t + \operatorname{sh} b_1 (T-t) e^{-bT}}{1 + e^{-2bT} - 2e^{-bT} \operatorname{ch} b_1 T}. \tag{1.39}$$

If $b^2 < \Omega^2$, $p_{1,2} = -b \pm i\Omega_1$, $\Omega_1 = \sqrt{\Omega^2 - b^2}$, then

$$\chi_1(t) = \frac{e^{-bt}}{\Omega_1} \frac{\sin \Omega_1 t + \sin \Omega_1 (T-t) e^{-bT}}{1 + e^{-2nT} - 2e^{-nT} \cos \Omega_1 T}. \tag{1.40}$$

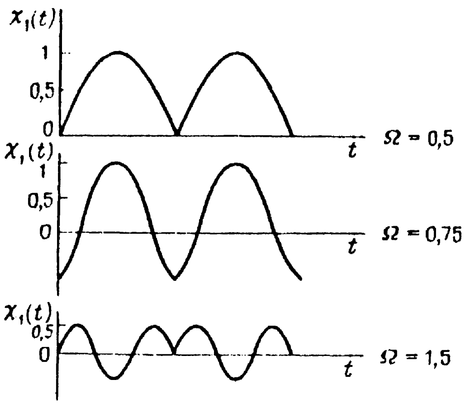


Fig. 1.4

In particular, if $b = 0$, then

$$\ddot{y} + \Omega^2 y = u,$$

$$\chi_1 = \frac{1}{2\Omega} \frac{\cos\Omega(t - T/2)}{\sin(\Omega T/2)},$$

$$0 < t < T \qquad (1.41)$$

The graphs of the function $\chi_1(t)$ are presented in Fig. 1.4 for $T = 2\pi$ and $\Omega = 0.5;\ 0.75;\ 1.5$.

Suppose that a system with n degrees of freedom has only oscillatory elements, i.e.,

$$L(p) = \prod_{r=1}^{n} \left(p^2 + \Omega_r^2\right),$$

$$M(p) = \beta_m \prod_{r=1}^{n} \left(p^2 + \gamma_q^2\right),$$

$$m \le n - 1.$$

Then the following presentation of the periodic Green's function of the oscillatory system results from (1.32) and (1.41):

$$\chi_1(t) = \beta_m \sum_{s=1}^{n} \frac{\prod_{q=1}^{m}\left(\gamma_q^2 - \Omega_s^2\right)}{2\Omega_s \prod_{r=1, r\ne s}^{n}{}^{*}\left(\Omega_r^2 - \Omega_s^2\right)} \frac{\cos[\Omega_s(t - T/2)]}{\sin(\Omega_s T/2)}, \qquad 0 < t < T/2. \qquad (1.42)$$

1.2.3 System's Response to the Excitation with the Half–Period Sign Change. Periodic Green's Function of the Second Kind

The case when a disturbance $u(t)$ has only odd harmonics

$$u(t) = \sum_{k=-\infty}^{\infty} u_{2k+1} e^{(2k+1)i\omega t} \qquad (1.43)$$

and changes its sign after the half-period

$$u(t + T/2) = -u(t), \qquad u(t) = u(t + T)$$

is of interest for applications. Substituting (1.43) into (1.26), a periodic response of the system to the disturbance of the type (1.43) can be expressed as

$$y_T(t)=\int_0^T \chi_1(t-s)u(s)ds=\int_0^{T/2}[\chi_1(t-s)-\chi_1(t-T/2-s)]u(s)ds\ . \qquad (1.44)$$

It follows from (1.27) that

$$\chi_1(t)-\chi_1(t-T/2)=\frac{2}{T}\sum_{k=-\infty}^{\infty}H[(2k+1)i\omega]\exp[(2k+1)i\omega t]=\chi_2(t). \qquad (1.45)$$

The function $\chi_2(t)$ is called *a periodic Green's function of the second kind* [115]. The equality

$$\chi_1(t-T/2)=\chi_1(t+T/2)$$

follows from the periodicity of $\chi_1(t)$. It means that

$$\chi_2(t)=\chi_1(t)-\chi_1(t-T/2)=\chi_1(t)-\chi_1(t+T/2). \qquad (1.46)$$

Thus, the periodic response to the disturbance $u(t)$ changing its sign after the half-period can be described by

$$y_T(t)=\int_0^{T/2}\chi_2(t-s)u(s)ds\ ,\ 0<t<T/2 \qquad (1.47)$$

It is obvious from the formal considerations, that the periodic Green's function of the second kind can be interpreted as the response of a linear system to the sequence of impulses, which change their signs after the half-period (Fig. 1.5):

$$\delta_2(t)=\delta^T(t)-\delta^T(t-T/2),$$

or, according to (2.25),

$$\delta_2^T(t)=\frac{2}{T}\sum_{k=-\infty}^{\infty}e^{(2k+1)i\omega t}\ . \qquad (1.48)$$

As was discussed in Section 1.2.2, the reduction of the periodic solution to the convolution (1.47) is expedient, when the kernel $\chi_2(t)$ can be presented in a closed form. Summing the series (1.45), we get

$$\chi_2(t)=\sum_{r=1}^{n}\frac{M(p_r)}{L'(p_r)}\frac{e^{p_r t}}{1-e^{p_r T/2}},\qquad 0<t<T/2\ , \qquad (1.49)$$

if the poles p_r of the function are simple, and

$$\chi_2=\sum_{i=1}^{l}\sum_{k=1}^{\nu_i}\frac{c_{ik}}{(k-1)!}\frac{d^{k-1}}{dp^{k-1}}\left[\frac{e^{pt}}{1-e^{pT/2}}\right]_{p=p_i},\qquad 0<t<T/2\ , \qquad (1.50)$$

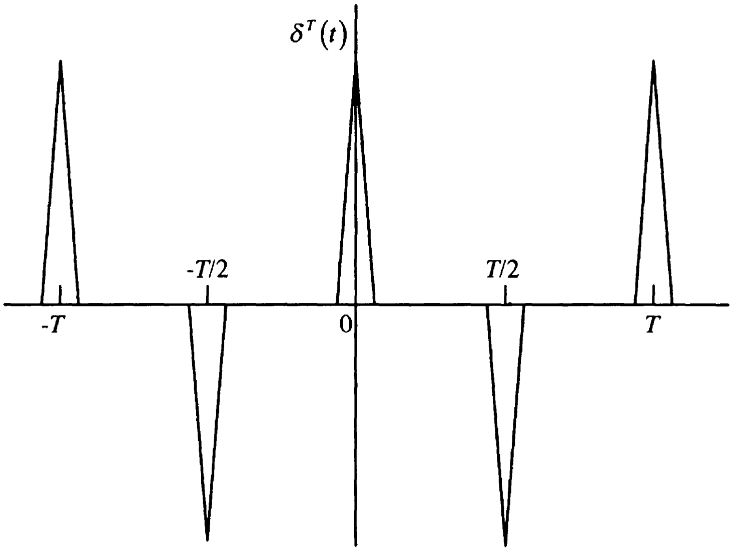


Fig. 1.5

if the poles are multiple. Coefficients c_{rk} in (1.50) have the same sense as in (1.34). For the periodic extension to $t > T/2$, the obvious property

$$\chi_2(t) = \chi_2(t+T),$$

$$\chi_2(t+T/2) = -\chi_2(t)$$

is used.

For a system with one degree of freedom

$$\ddot{y} + 2b\dot{y} + \Omega^2 y = u$$

the periodic Green's function of the second kind can be presented in the form, analogous to (1.39)-(1.41):

for $b^2 > \Omega^2$, $b_1^2 = b^2 - \Omega^2$

$$\chi_2(t) = \frac{e^{-bt}}{b_1} \frac{\operatorname{sh} b_1 t + e^{-bT/2} \operatorname{sh} b_1 (t - T/2)}{1 + e^{-bT} - 2e^{-bT/2} \operatorname{ch} b_1 T}, \tag{1.51}$$

for $b^2 < \Omega^2$, $b_1^2 = b^2 - \Omega^2$

$$\chi_2(t) = \frac{e^{-bt}}{b_1} \frac{\sin \Omega_1 t + e^{-bT/2} \sin \Omega_1 (t - T/2)}{1 + e^{-bT} - 2e^{-bT/2} \cos \Omega_1 T} \tag{1.52}$$

for $b = 0$

$$\chi_2 = \frac{1}{2\Omega} \frac{\sin \Omega (t - T/4)}{\cos(\Omega T/4)}, \qquad 0 < t < T/2 \,. \tag{1.53}$$

For a system, containing only oscillatory elements,

$$L(p) = \prod_{r=1}^{n} \left(p^2 + \Omega_r^2\right), \qquad M(p) = \beta_m \prod_{r=1}^{n} \left(p^2 + \gamma_q^2\right), \qquad m \le n-1,$$

we have in analogy with (1.42)

$$\chi_2(t) = \beta_m \sum_{s=1}^{n} \frac{\prod_{q=1}^{m} \left(\gamma_q^2 - \Omega_s^2\right)}{2\Omega_s \prod_{r=1, r\neq s}^{n}{}^{*} \left(\Omega_r^2 - \Omega_s^2\right)} \frac{\sin[\Omega_s (t - T/4)]}{\cos(\Omega_s T/4)}, \tag{1.54}$$

$0 < t < T/4$.

1.2.4 Integral Equations of Periodic Oscillations of Non-linear Systems

The method of integral equations can be also expanded to systems with non-linear elements. For the sake of briefness let us limit considerations to the case of the system, dynamics of which is described by the equation

$$D(p)x = \mu d(p)[g(t,x)] + R(p)u \,. \tag{1.55}$$

Here $D(p)$ is an operator of dynamic stiffness of the linear part of the system, $R(p)$ is a transfer function of the control, the function $g(t,x)$ and operator $d(p)$ reflect the effects of non-stationary and non-linear links in the system, μ is a parameter.

Let $u(t)$ and $g(t,x)$ be the T-periodic excitations, generating in the system (1.55) a T-periodic regime. Supposing, that the system is non-resonant, i.e.,

$$D(ki\omega) \neq 0, \qquad R^{-1}(ki\omega) \neq 0, \qquad d^{-1}(ki\omega) \neq 0$$

for $\omega = 2\pi / T$, $k = 0, \pm 1, \pm 2, \ldots$ Using the same scheme as in the case of the linear system, let us write the integral equation of the periodic regime in the form

$$x(t) = \int_0^T \chi_1(t-s) u(s) ds + \mu \int_0^T \chi_1^g (t-s) g(s, x(s)) ds \,, \tag{1.56}$$

where χ_1, χ_1^g are periodic Green's functions of respective elements of the system

$$\chi_1(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}D^{-1}(ki\omega)R(ki\omega)e^{ki\omega t} ,$$

$$\chi_1^g(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}D^{-1}(ki\omega)d(ki\omega)e^{ki\omega t} . \tag{1.57}$$

An attempt to obtain an analytical solution for a non-linear problem fails, as a rule. The existence conditions for the T-periodic regime and approximated methods of solution of equations (1.55) are discussed in [115].

1.3 Necessary Conditions of Optimality for Periodic Regimes

We now consider the systems, dynamics of which is described by equations of the type (1.54). It is necessary to find a piecewise continuous periodic control $u(t)=u(t+T)$ minimizing some quality criterion for T-periodic trajectories of the system. Periodicity conditions allow us to describe the system movement by an integral equation of the type (1.56), or, in a more general case, by the equation

$$x(t)=\int_0^T \chi_1(t-s)f(s,x(s),u(s),a)ds , \tag{1.58}$$

$$0<t<T , \qquad x,u\in R_1 .$$

The control $u(t)$ is obtained from the minimum condition of the functional

$$\Phi(u)=\int_0^T f_0(t,x(t),u(t),a)dt+F(T,x(T),a) , \tag{1.59}$$

Here $\chi_1(t-s)$ is the periodic Green's function of the linear stationary part of the system, a is an unknown parameter which is determined mutually with $u(t)$ from the optimality conditions.

Eq. (1.58) is treated as a constraint linking the trajectory and control. Besides (1.58), another constraints can exist, for instance,

a) isoperimeter links

$$g_1^\nu=\int_0^T f_1^\nu(t,x(t),u(t),a)dt=0 , \qquad \nu=1,\dots,r ; \tag{1.60}$$

b) point constraints

$$g_2^\nu=f_2\left[t_\nu,x(t_\nu)\right]=0 , \qquad 0<t_\nu<T , \qquad \nu=1,\dots,m . \tag{1.61}$$

If such a constraint is given at the end of the interval, then either $t_0=0_+$ or

$t_m = T_-$ is considered; it means that the point t_0 remains an inner point of the interval $(0,T)$.

The value of the function F for the inner point of the interval can be also minimized, that means, the control $u(t)$ minimizing the functional

$$\Phi(u) = F\left(t_0, x(t_0), a\right), \quad 0 < t_0 < T \tag{1.62}$$

can be found.

As above, for the function at the end of the interval, either $t_0 = 0_+$ or $t_0 = T_-$ should be considered.

In addition to the conditions of integral type (1.60), there exit control constraints of the form $u \in U$, where U is a region of the space of respective dimension. These constraints for systems with a scalar control usually have the form $U_1 \le u \le U_2$.

The solution of the problem (1.58)-(1.62) belongs to the set of admissible controls. Let us call the function $\tilde{u}(t)$ *the admissible control*, if $\tilde{u}(t)$ is a piecewise continuous, T-periodic function, which satisfies the condition $\tilde{u}(t) \in U$ for all $t \in [0,T]$, and if for $u = \tilde{u}(t)$ there exists a unique solution of Eq. (1.58).

It should be remembered, that Eq. (1.58) is an equivalent form of respective differential motion equations and periodicity conditions. Thus, the existence problem for optimal periodic controls is solved in the same way, as for the equivalent system of differential equations [63, 123, 124, 187]. Below it is always considered, that the optimal control exists and it is determined by equations of the maximum principle.

Conditions of the maximum principles can be obtained by re-writing the optimality conditions of a periodic control for differential equations (see Section A.4) in the form of equivalent integral equations. Still, it is expedient to obtain this relations direct for the problem (1.58)-(1.62).

Various variants of the maximum principle are formulated in [35, 133, 140-142, 152] for problems with constraints in the form of integral equations. As usual, different variants of equations (Volterra [140-142], Fredholm [140], Hammerstein [198], Uryson [143]) and of functionals and constraints were studied separately. The problem was treated in the most general formulation in [162, 193, 194]: The minimum conditions for the functional (1.59) with constraints in the form of (1.58), (1.60), (1.61) were obtained.

The obvious procedure for obtaining of the necessary optimality conditions for problems with diverse constraints is suggested in [133]. The equations of the maximum principle, obtained in [133], coincide with the conclusions of the works [162, 193, 194]. Therefore, the approach of [133] would be used for the explanation of the conditions of the maximum principle.

Let us write all the constraints of the problem in the form of isoperimeter links. Premultiplying the left-hand and right-hand parts of Eq. (1.58) and integrating for the interval (0,*T*), we get

$$g_0 = \int_0^T p(t)\left[x(t)ds - \int_0^T \chi_1(t-s) f(t,s,x(s),u(s),a)ds\right]dt =$$

$$= \int_0^T \left\{p(t) - \int_0^T p(s)\chi_1(s-t) f(s,t,x(t),u(t),a)ds\right\}dt. \qquad (1.63)$$

Isoperimeter links (1.60) can be written in the form

$$\sum_{\nu=1}^{r} \lambda_1^\nu g_1^\nu = \sum_{\nu=1}^{r} \lambda_1^\nu \int_0^T f_1^\nu(t,x(t),u(t),a)dt = 0, \qquad (1.64)$$

where $\lambda_1^\nu \neq 0$ are the arbitrary multipliers.

The constraint (1.64) can be reduced to (1.63) with the help of the Dirac delta function

$$g_2^\nu = f_2^\nu(t_\nu, x(t_\nu),) = \int_0^T f_2^\nu(t,x(t))\delta(t-t_\nu)ds , \qquad (1.65)$$

for constraints in finite points of the interval it should be considered

$$f_2(0,x(0)) = \int_0^T f_2(t,x(t))\delta(t-0_+)dt = f_2(0_+,x(0_+)) , \qquad (1.66)$$

$$f_2(T,x(T)) = \int_0^T f_2(t,x(t))\delta(t-T_-)dt = f_2(T_-,x(T_-)) .$$

Then the constraints of the type (1.60), (1.66) can be written in the form, analogous to (1.64)

$$\sum_{\nu=1}^{r} \lambda_2^\nu \int_0^T f_2^\nu(t,x(t))\delta(t-t_\nu)dt = 0, \qquad (1.67)$$

where $\lambda_2^\nu \neq 0$ are the arbitrary multipliers.

Finally, the functional (1.59) can also be written in the integral form

$$\int_0^T \left[f_0(t,x(t),u(t),a) + F(t,x(t))\delta(t-T_-)\right]dt . \qquad (1.68)$$

Premultiplying (1.68) with a constant λ_0 and adding the constraints equations (1.63), (1.64), (1.67), we get

$$S = \lambda_0\Phi + \sum_{\nu=1}^{r} \lambda_1^\nu g_1^\nu + \sum_{\nu=1}^{r} \lambda_2^\nu g_2^\nu + g_0 = \int_0^T L(t,x(t),u(t),a,p(t),\lambda)dt , \qquad (1.69)$$

where

$$L(t,x,u,a,p,\lambda)=\lambda_0\left[f_0(t,x,u,a)+F(t,x)\delta(t-T_-)\right]+$$

$$+\sum_{\nu=1}^{r}\lambda_1^\nu f_1^\nu(t,x,u,a)+\sum_{\nu=1}^{m}\lambda_2^\nu f_2^\nu(t,x)\delta(t-t_\nu)+p(t)x(t)-$$

$$-\int_0^T p(s)\chi_1(s-t)f(s,t,x(t),u(t),a)ds \ . \tag{1.70}$$

Eq. (1.69) is called a *Lagrange functional*, and the function (1.70) - a *Lagrange function*. Extremum conditions for the functional F (or S) can be obtained by means of standard considerations [31, 169]. Let $u(t)$ denote some continuous periodic control, $x(t)$ – respective trajectory, and $\delta u(t)$ – arbitrary continuous control variation. If the admissible control domain is not bounded, then the control $u_\varepsilon = u(t)+\varepsilon\delta u(t)$ also belongs to the class of admissible ones. A periodic solution $x_\varepsilon(t)$. corresponds to the control $u_\varepsilon(t)$. Suppose that functions f_0, $f_{1,2}^\nu$, f are continuously differentiable with respect to their arguments, while the period T and parameter α are fixed. The functional variation which corresponds to the control variation is

$$\delta S=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left[S(u_\varepsilon)-S(u)\right]=\int_0^T\left[L_x(t,x,u,a,p,\lambda)\delta x-L_u(t,x,u,a,p,\lambda)\delta u\right]dt\ . \tag{1.71}$$

Here $\delta x=\left[\partial x_\varepsilon/\partial\varepsilon\right]_{\varepsilon=0}$ is the trajectory variation.

If the multipliers $p(t)$, λ_1^ν, λ_2^ν are chosen so, that the coefficient by δx reduces to zero,

$$\partial L/\partial x = 0\ , \tag{1.72}$$

then the functional variation gets the form

$$\delta S=\int_0^T L_u(t,....)\delta u dt = 0\ .$$

If S reaches the extremum value, then $\delta S=0$ for arbitrary variations δu, and according to the main lemma of the variational calculus

$$\partial L/\partial u = 0\ . \tag{1.73}$$

Thus, we have one dynamic equation (1.58), $r+m$ constraint equations (1.60), (1.61) and two stationary conditions (1.72), (1.73) for the determination of the unknown values of $x(t), u(t)$, r unknown values of λ_1^ν, m unknown values of λ_2^ν and of multiplier $p(t)$.

If $x(t)\in R_n$ is an n-dimensional vector satisfying the equation

$$x(t)=\int_0^T K(t-s)f(t,s,x(s),u(s),a)ds\,, \tag{1.74}$$

where a kernel $K(t-s)$ is a matrix periodic Green's function, $u\in R_m$, $f\in R_n$, then obtaining the scalar product of left and right parts of Eq. (1.74) with n-dimensional vector of Lagrange multipliers $P(t)$, we get the constraint equation (1.63) in the form

$$g_0=\int_0^T\left\{P'(t)x(t)-\int_0^T P'(s)K(s-t)f(s,t,x(t),u(t),a)ds\right\}dt\,.$$

In the same way we can reduce the function L; conditions (1.72), (1.73) hold; but Eq. (1.72) reduces to the system of n equations. If the constraint equations contain the unknown parameter a which is determined by the optimality conditions, then the variation of the functional (1.71) gets the form

$$\delta S=\int_0^T\left[\frac{\partial L}{\partial x}\delta x-\frac{\partial L}{\partial u}\delta u\right]dt+\frac{\partial S}{\partial a}\delta a\,, \tag{1.75}$$

and the conditions (1.72), (1.73) are extended by a supplement

$$\frac{\partial S}{\partial a}\delta a\le 0\,, \tag{1.76}$$

where δa is an admissible variation of the parameter a. If the set of admissible values a is open, and the integration bounds do not depend on it, then Eq. (1.76) is reduced to

$$\int_0^T\frac{\partial L}{\partial a}dt=0\,. \tag{1.77}$$

If the integration bounds depend on a, but the set of a values is open, then Eq. (1.76) can be reduced to the form

$$L(T,x(T),u(T),a,P(T),\lambda)\frac{\partial T}{\partial a}+\int_0^T\frac{\partial}{\partial x}L(t,x(t),u(t),a,P(t),\lambda)dt=0\,. \tag{1.78}$$

In particular, if the period T is unknown, $a\equiv T$, then Eq. (1.78) gets the form

$$L(t,x,u,a,P,\lambda)\big|_{t=T}+\int_0^T\frac{\partial}{\partial x}L(t,x(t),u(t),T_-,P(t),\lambda)dt=0\,. \tag{1.79}$$

where x, u are calculated with account for (1.72), (1.73).

For obtaining of (1.72), (1.73) the admissible control domain was considered to be non-restricted and the controls to be continuous. In a general case the following *maximum principle* holds [133, 162, 193, 194].

Theorem 3.1. *Let*

1) *for* $x \in R_n$ $u \in U \subset R_m$, $a \in A$, $t \in (0,T)$ *the functions f,* f_0, f_1^ν, f_2^ν *be continuous for all variables together with derivatives with respect to x, a;*

2) *the optimal control* u_*, *corresponding to it optimal trajectory* x_* *and optimal value of parameter* a_* *exist.*

Then there exists the scalar $\lambda_0 \le 0$, *constant vectors* $\lambda_1 = [\lambda_1^1, \dots, \lambda_1^r]$, $\lambda_2 = [\lambda_2^1, \dots, \lambda_2^m]$ *and the solution of equation (1.72) vector* $P(t)$, *which is piecewise continuous with respect to t and contains a finite number of singular components in form of* δ *-functions, such, that*

$$L(t, x_*, u_*, a_*, P, \lambda) = \max_{u \in U} L(t, x_*, u_*, a_*, P, \lambda) \tag{1.80}$$

and

$$\left.\frac{\partial S}{\partial a}\right|_{a=a_*} \delta a \le 0, \tag{1.81}$$

for each variation $\delta a \in A$ *such that* $a_* + \delta a \in A$.

We will next show the relation between the equations of the maximum principle for periodic control (A.4) and Eqs. (1.72), (1.80).

Consider a system of equations with the main linear part

$$\dot{x} = Ax + f(t, x, u)), \quad x \in R_n, \quad u \in R_m, \tag{1.82}$$

where $f(t, x, u)$ is a T-periodic function of t, satisfying the necessary conditions of smoothness with respect to t, x, u (A.4). Suppose also that the eigenvalues of the matrix A differ from $\pm i2\pi k / T$, $k = 0$, $1, \dots$ (non-resonant case). Let us find the control $u(t)$, minimizing the functional

$$\Phi(u) = \int_0^T f(t, x, u) dt \, / \, u \in U \tag{1.83}$$

for a T-periodic solution of (1.82). The function f_0 is supposed to be T-periodic with respect to t and sufficiently smooth with respect to all arguments, U-compact in R_m. The Hamilton function of the problem (1.82), (1.83) has the form

$$H(t, x, u, q) = q'[Ax + f(t, x, u)] - f_0(t, x, u), \tag{1.84}$$

where $q(t)$ is a T-periodic solution of the equation

$$\dot{q} = f_{0x}(t, x, u) - Aq' - f_x'(t, x, u) q. \tag{1.85}$$

The optimal control is determined by the maximum principle

$$u = \arg\max_{u \in U}\left[q' f(t,x,u) - f_0(t,x,u)\right]. \tag{1.86}$$

Introduce a new variable

$$P = \dot{q} + A'q\,, \tag{1.87}$$

or, according to (1.85),

$$P = f_{0x}(t,x,u) - f_x'(t,x,u)q\,. \tag{1.88}$$

Accounting for the periodicity condition, let us re-write (1.82), (1.87) in the form of integral equations

$$x(t) = -\int_0^T K(t-s) f(s,x,u)ds\,, \tag{1.89}$$

where

$$K(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}(ki\omega I - A)^{-1} e^{ki\omega t} \tag{1.90}$$

is a matrix periodic Green's function of the linear part of the system (1.82). In analogy with

$$q(t) = -\int_0^T K'(s-t)P(s)ds\,. \tag{1.91}$$

Accounting for (1.87), (1.91), we can reformulate Eqs. (1.86), (1.88) to the form

$$u = \arg\max_{u \in U} Q(t,x,u,P), \tag{1.92}$$

$$Q = -f_0(t,x,u) - \int_0^T P'(s)K(s-t)ds f(t,x,u),$$

$$P(t) = -f_{0x}'(t,x,u) + \int_0^T K'(s-t)P(s)ds f_x(t,x,u). \tag{1.93}$$

It is obvious, that the expression

$$L(t,x,u,P) = P'tx(t) + Q(t,x,u,P) \tag{1.94}$$

is the expanded Lagrange function of the problem (1.89), (1.93), and the Lagrange multiplier P satisfies the equation

$$\partial L/\partial x = 0\,. \tag{1.95}$$

Thus, the periodic optimization problem (1.83) – (1.86) is equivalent to the problem (1.83), (1.89), (1.92) – (1.95). The Lagrange multipliers q and P are linked by equations (1.87) and (1.91).

Suppose that only the j-th component f_j of the vector f is other than zero and that the function u is a scalar one. The functions f_j and f_0 depend only on one, for instance, on the first component of the vector $f_j = f_j(t, x_1, u)$, $f_0 = f_0(t, x_1, u)$. Then, it is expedient to single out the scalar equation for the component x_1. We have from (1.89)

$$x_1(t) = \int_0^T \chi_{1j}(t-s) f_j(s, x_1, u) ds. \tag{1.96}$$

Here χ_{1j} is the element $(1, j)$ of the matrix K.

Further, from (1.88) we have

$$P = (p, 0, \ldots, 0), \tag{1.97}$$

where

$$\dot{p} = f_{0x_1}(t, x_1, u) - q_j f_{jx_1}(t, x_1, u), \tag{1.98}$$

q_j is the j-th component of the vector q. Correspondingly, from Eq. (1.97)

$$Q = -f_0(t, x_1, u) - \int_0^T p(s) \chi_{1j}(s-t) ds f_j(t, x_1, u), \tag{1.99}$$

$$L = p(t) x_1(t) - \int_0^T p(s) \chi_{1j}(s-t) ds f_j(t, x_1, u) - f_0(t, x_1, u) \tag{1.100}$$

[compare with (1.70)].

The equivalence of the form for the periodic control problems can be stated analogously in the form of differential and integral equations, and in the more general case in the form of the functional (1.59) with the constraints (1.60), (1.61).

Periodic regimes are the partial case of the stationary regimes, restricted over the entire axis. Some optimization problems for nearly-periodic and stationary regimes are analyzed in [10, 11, 25, 40, 41, 89, 159].

1.4 Optimal Periodic Control for Linear Systems (Non-resonant Case)

Considerable difficulties by solution of applied problems are connected with the choice of an optimized functional. Criteria which characterize the system quality are determined by its purpose: In problems of optimal displacement it is a high-speed action, as a rule; in vibroprotection problems they are the level of displacements or of absolute accelerations of certain points, etc. At the same time, control constraints are usually linked with the realization method for controlling excitations.

We will next examine some traditional formulations for problems of the optimal periodic control.

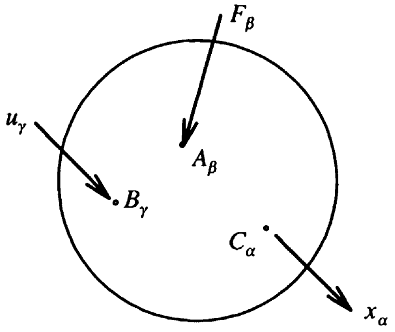


Fig. 1. 6

Let external excitations $F_1,\ldots,F_k$ be applied to the points $A_1, A_2,\ldots, A_k$, and controlling excitations $u_1,\ldots,u_k$ be applied to the points $B_1,\ldots, B_k$ (Fig. 1.6). It is necessary to find such control excitations $u_j(t)$, that responses $C_1, C_2,\ldots$, which are characterized by displacements of certain points of the object, should not exceed admissible values and the control costs should be minimal.

Let $x=(x_1,\ldots,x_r)$ be a r-dimensional displacement vector of the object's points of interest, $f=(F_1, \ldots, F_k)$ be a k-dimensional excitation vector, $u=(u_1,\ldots,u_m)$ be a m-dimensional control vector, so that

$$x=E_1(p)f+E_u(p)u, \tag{1.101}$$

where $E_1(p)$ and $E_u(p)$ are matrixes of dynamic compliance, $f(t)=f(t+T)$. A periodic control $u(t)=u(t+T)$ can be found from the minimum condition for the functional

$$\Phi(u)=\frac{1}{T}\int_0^T(x'R_1x+u'R_2u)dt, \tag{1.102}$$

where $R_1 \geq 0$ and $R_2>0$ are symmetric matrixes, a prime as a superscript denotes transpose, and the period T is considered to be given.

In order to use the results of Section 1.3, let us write the integral equation of the periodic regime

$$x(t)=\int_0^T\left[K_f(t-s)f(s)+K_u(t-s)u(s)\right]ds, \tag{1.103}$$

where K_f and K_u are the matrix periodic Green's function of the first kind, corresponding to the operators E_f and E_u:

$$K(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}E(ki\omega)e^{ki\omega t}. \tag{1.104}$$

The Lagrange function (1.70) for the problem (1.102), (1.103) can be written in the form

$$L(s,x,u,P) = -\frac{1}{2}\left[x'(s)R_1x(s) + u'(s)R_2u(s)\right] + P'(s)x(s) - $$
$$-\int_0^T P'(t)\left[K_f(t-s)f(s) + K_u(t-s)u(s)\right]dt,$$

where $P(t)$ is the vector of Lagrange multipliers. The stationary conditions $\partial L/\partial u = 0$, $\partial L/\partial x = 0$ give

$$-R_2u(s) - \int_0^T K_u'(t-s)P(t)dt = 0,$$
$$-R_1x(s) + P(s) = 0.$$

Excluding $P(s)$, we get the integral relation

$$u(s) = -R_2^{-1}\int_0^T K_u'(t-s)R_1x(t)dt, \tag{1.105}$$

which can be written in the operator form

$$u = -W(p)x, \quad W(p) = R_2^{-1}E_u'(-p)R_1. \tag{1.106}$$

Here the operator $W(p)$ is understood not as a transfer function of the feedback, but as the shortened form of the convolution transform. Eq. (1.106) only links periodic components of the functions $u(t)$ and $x(t)$: if

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{ki\omega t}, \text{ then}$$

$$u = \sum_{k=-\infty}^{\infty} u_k e^{ki\omega t}, \quad u_k = -W(ki\omega)x_k. \tag{1.107}$$

Substituting (1.106) into (1.101), we get

$$x = \left[I + E_u(p)W(p)\right]^{-1}E_f(p)f, \tag{1.108}$$
$$u = -W(p)\left[I + E_u(p)W(p)\right]^{-1}E_f(p)f.$$

If $f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ki\omega t}$, then we have the equation for the optimal periodic control and for respective optimal trajectory

$$u_*(t) = -\sum_{k=-\infty}^{\infty} W(ki\omega)\left[I + E_u(ki\omega)W(ki\omega)\right]^{-1}E_f(ki\omega)f_k e^{ki\omega t}, \tag{1.109}$$

$$x_*(t) = \sum_{k=-\infty}^{\infty}\left[I + E_u(ki\omega)W(ki\omega)\right]^{-1}E_f(ki\omega)f_k e^{ki\omega t}$$

directly from (1.107).

Obvious results can be obtained for a one-dimensional system

$$x = l_f(p)f + l_u(p)u . \tag{1.110}$$

In the functional (1.102) $R_1 = 1$ and $R_2 = r$ should be considered:

$$\Phi(u) = \frac{1}{T}\int_0^T \left(x^2 + ru^2\right)dt . \tag{1.111}$$

Then

$$u = -W(p)x , \quad W(p) = r^{-1}l_u(-p), \tag{1.112}$$

$$x = \left[1 + r^{-1}l_u(p)l_u(-p)\right]^{-1} l_f(p)f). \tag{1.113}$$

The minimum value of the functional (1.102) is easy to calculate with the help of Eq. (1.109). Let

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ki\omega t} , \quad |f_k| = |f_{-k}| .$$

Then the periodic trajectory and periodic control have the form analogous to (1.105)

$$x(t) = \sum_{k=-\infty}^{\infty} \left[1 + r^{-1}\left|l_u(ki\omega)\right|^2\right]^{-1} l_f(ki\omega) f_k e^{ki\omega t} , \tag{1.114}$$

$$u(t) = -\sum_{k=-\infty}^{\infty} r^{-1} l_u(ki\omega)\left[1 + r^{-1}\left|l_u(ki\omega)\right|^2\right]^{-1} l_f(ki\omega) f_k e^{ki\omega t} .$$

Substituting Eq. (1.114) into (1.111), we get

$$\Phi(u_*) = \left[1 + r^{-1}\left|l_u(0)\right|^2\right]^{-1}\left|l_f(0)\right|^2 f_0^2 + 2\sum_{k=1}^{\infty}\left[1 + r^{-1}\left|l_u(ki\omega)\right|^2\right]^{-1}\left|l_f(ki\omega)\right|^2 f_k^2 , \tag{1.115}$$

and for harmonic excitation $f(t) = f\cos(\omega t) + \varphi$

$$\Phi(u_*) = \left[1 + r^{-1}\left|l_u(i\omega)\right|^2\right]^{-1}\left|l_f(i\omega)\right|^2 f . \tag{1.116}$$

In real systems the amplitudes of harmonic components sharply decrease with the increase in the harmonic's number, so it makes sense to diminish the level only of some first harmonics. This decreases control costs.

Let us once again examine the system (1.101) with the quality functional, other than in (1.102)

$$\Phi(u) = \frac{1}{T}\int_0^T \left(x_l' R_1 x_l + u' R_2 u\right)dt , \tag{1.117}$$

where x_l is the sum of the first m harmonics of the solution (*low-frequency component* [115])

$$x_l = \sum_{k\in M_l} x_k e^{ki\omega t}, \qquad M_l = [-m, m]. \tag{1.118}$$

Let us single out the low-frequency components in periodic Green's functions K_f, K_u. Then we have

$$K_{vl} = \frac{1}{T}\sum_{k\in M_l} E_v(ki\omega)e^{ki\omega t}. \qquad v \to f, u. \tag{1.119}$$

It is not complicate to prove the correctness of the relation

$$x_l(t) = \frac{1}{T}\int_0^T [K_{fl}(t-s)f(s) + K_{ul}(t-s)u(s)]ds. \tag{1.120}$$

Thus, the control problem for low-frequency components can be singled out in a linear system. The problem (1.120), (1.117) has the structure which is equivalent to the one of Eqs. (1.102), (1.103) with replacement of the periodic Green's functions (1.104) by their low-frequency components. So, we can directly write the relations for the optimal control and optimal trajectory

$$u_*(t) = \sum_{k\in M_l} W(ki\omega)[I + E_u(ki\omega)W(ki\omega)]^{-1} E_f(ki\omega) f_k e^{ki\omega t}, \tag{1.121}$$

$$x_{*l}(t) = \sum_{k\in M_l} [I + E_u(ki\omega)W(ki\omega)]^{-1} E_f(ki\omega) f_k e^{ki\omega t}. \tag{1.122}$$

In particular, only one resonance harmonica can be controlled, considering the oscillation level outside the resonance sufficiently small.

If control for the system (1.101) is formed with the use of a feedback system, then the cost estimation by the mean–square criterion has no strict physical meaning. It follows from Eq. (1.106), that in this case the criterion (1.102) can be interpreted in the following way: The control $u(t)$ which is being realized by a linear feedback and which minimizes some mean-square quality criterion should be found. Weight coefficients R_1, R_2 should be chosen with account for structural constraints.

If we have the problem of the construction of an optimal control $u(t) = u_0 + \sum_{k=1}^{\infty} u_k \cos(k\omega t - \varphi_k)$, then the value

$$\Phi(u) = \frac{1}{T}\int_0^T u^2 dt = u_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} u_k^2,$$

usually characterizes the control energy costs. Thus, when the controlling excitation is generated by an electric motor, and $u(t)$ is proportional to the strength of the current, then the functional Φ is proportional to the power of heat loss [15]; if $u(t)$ is a torque on an input shaft of the controlled mechanism, then Φ characterizes the dynamic load level of the shaft [36], etc.

There is a close connection between the problem of the optimal control construction and the problem of synthesis and practical realization of the control in the feedback form. Obviously, for the harmonic excitation or the excitation containing a finite number of harmonics, it is always possible to construct the control of the form $u=-V(p)x$ with a transfer function $V(p)$ coinciding for the given frequency ω with the optimal one:

$$V(i\omega)=r^{-1}l_u(-i\omega). \tag{1.123}$$

Then the solution of the system

$$[1+l_u(p)V(p)]x=l_f(p)f \tag{1.124}$$

for the given frequency also coincides with the optimal one. The method for the optimal synthesis realization of some control systems is shown in [25, 36].

Another approach to construction of optimal systems is also possible. Eq. (1.114) can be interpreted as the utmost admissible control with the corresponding trajectory. If the control in a real system differs form the optimal one, then, calculating the corresponding trajectory and comparing the value of the quality criterion with utmost admissible one, we get the effectiveness estimate of the control for the given criterion.

Example. Let us examine a stabilization problem of the angular velocity of the mechanism in order to illustrate the introduced concepts. The simplest scheme of the mechanism with stiff elements is presented in Fig. 1.7.

Periodic excitations $F(t)$ are linked with variability of reduced insertion moments of the motor M and actuator A. The period of excitations is $T=2\pi/\omega$, where ω is the frequency of the programmed movement. Excitations $F(t)$ generate movement deviations from the programmed movement; they are characterized by a dynamic error $\xi(t)$. A movement stabilization is carried out by application of the controlling moment $u(t)$ to the output shaft. The disturbed movement equation has the form [36]

$$J_0\ddot{\xi}+\beta\dot{\xi}-\mu=F(t)+u(t), \tag{1.125}$$

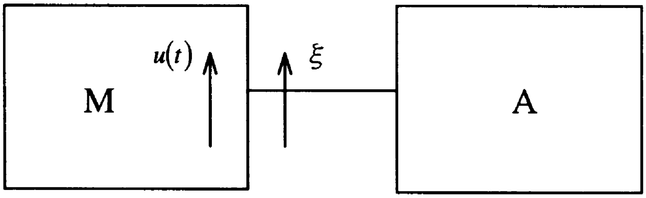


Fig. 1.7

where J_0 is a constant component of the reduced insertion moment, β is a characteristic of the resistance moment, μ is the ratio of the output moment of the motor from the stationary one, corresponding to the uniform rotation regime. Accounting for motor characteristics [36],

$$\tau\dot{\mu} + \mu + \alpha\dot{\xi} = 0\,, \tag{1.126}$$

where τ, α are motor parameters.

In the problem of the angular velocity stabilization with control constraints, the minimized functional has the form

$$\Phi(u) = \frac{1}{T}\int_0^T \left(\dot{\xi}^2 + ru^2\right)dt\,. \tag{1.127}$$

Introducing a new variable $v = \dot{\xi}$ and writing with account for (1.126)

$$\mu = -\alpha(\tau p + 1)^{-1} v\,, \tag{1.128}$$

we get

$$\left[J_0 p(\tau p + 1) + \beta(\tau p + 1) + \alpha\right]v = (\tau p + 1)(F + u). \tag{1.129}$$

It is obvious, that

$$l_u(p) = l_f(p) = (1 + \tau p)\left[J_0 \tau p^2 + (J_0 + \tau\beta)p + (\alpha + \beta)\right]^{-1}. \tag{1.130}$$

Suppose that the problem consists only of the system stabilization with respect to the first harmonic and $F(t) = f\cos(\omega t + \varphi)$. Then

$$l_u(i\omega) = l_f(i\omega) = A - i\omega B\,,$$
$$A = D^{-1}(\omega)\left[\alpha + \beta\left(1 + \tau^2\omega^2\right)\right] > 0\,,$$

(1.131)

$$B = D^{-1}(\omega)\left[J_0\left(1+\tau^2\omega^2\right)-\alpha\tau\right],$$
$$D(\omega) = \left(\alpha+\beta-J_0\tau\omega^2\right)^2+\omega^2\left(\tau\beta+J_0\right)^2,$$

and $B > 0$ for sufficiently small τ.

Let us realize a feedback with respect to v, which forms the optimal control for a fixed frequency ω. We look for a solution in the form $u = -V(p)v$, for a control, optimal according to (1.123), (1.131) for the frequency ω:

$$V(i\omega) = r^{-1}l_u(-i\omega) = A + i\omega B. \tag{1.132}$$

One of the most widely-spread stabilization methods is an attaching of a fly-wheel to a motor shaft, that results in an additional moment $M_u = -J_M\ddot{\xi} = -J_M pv$, where J_M is a moment of the fly-wheel. In other words,

$$V(p) = J_M p, \quad V(i\omega) = i\omega J_M. \tag{1.133}$$

Comparing (1.133) and (1.132), we see that it is impossible to achieve the coincidence of Eqs. (1.132), (1.133) for any value of J_M. In other words, the fly-wheel allows the diminishing of rotation non-uniformity, but the quality criterion value exceeds the optimal one (a detailed analysis of this problem for more complicated schemes of mechanisms is studied in [36]).

Another method of rotation stabilization is the use of the damper of rotatory oscillations, generating the moment [36]

$$M_u = -\frac{\rho h p}{\rho p + h} = -V(p)v. \tag{1.134}$$

Here ρ is the damper mass, h is the damping coefficient. Thus,

$$V(i\omega) = \frac{i\omega\rho h}{\rho i\omega + h}\frac{\rho^2 h\omega^2 + i\omega\rho h^2}{(\rho\omega)^2 + h^2}. \tag{1.135}$$

Comparing (1.132) to (1.135), note that choosing ρ and h from conditions

$$\rho h\omega^2\left[(\rho\omega)^2+h^2\right]^{-1} = A,$$
$$\rho h^2\left[(\rho\omega)^2+h^2\right]^{-1} = B, \tag{1.136}$$

we get the control $u = M_u$, which realizes the optimal value of the functional for the given excitation frequency.

Another widely-spread variant of the problem is determination of the control $u(t)$ which minimizes the functional

$$\Phi(u) = x'(t_*)R_3x(t_*) + \frac{1}{T}\int_0^T u_2'(t)R_2u(t)dt, \tag{1.137}$$

at trajectories of the system (1.141). Here $t_* \in (0,T)$ is the fixed time, $R_3 \geq 0$ is a symmetric matrix.

According to Eq. (1.70), the Lagrange function of the problem has the form

$$L(s,x,u,P) = -\frac{1}{2}[x'(s)R_3x(s)\delta(s-t_*) + u'(s)R_2u(s)] + P'(s)x(s) -$$
$$-\int_0^T P'(t)[K_f(t-s)f(s) + K_u(t-s)u(s)]dt, \tag{1.138}$$

where $P(t)$ is the vector of Lagrange multipliers.

Stationary conditions lead to the following equations for determination of $u(s)$, $P(s)$:

$$-R_2u(s) + \int_0^T K_u(t-s)P(t)dt = 0,$$
$$-R_3x(s)\delta(s-t_*) + P(s) = 0,$$

i.e.,

$$u(s) = -R_2^{-1}\int_0^T K_u(t-s)R_3x(t)\delta(t-t_*)dt = -R_2^{-1}K_u(t_*-s)R_3x(t_*). \tag{1.139}$$

If the point t_* is not fixed, and is determined from the condition

$$x'(t_*)R_3x(t_*) = \max x'(t)R_3x(t), \quad 0 < t_* < T, \tag{1.140}$$

then the value t_* can be treated as a parameter which is determined by the optimality condition. According to (1.77) we have

$$\frac{\partial}{\partial t_*}\int_0^T L(s,\ldots)ds = 0. \tag{1.141}$$

Owing to Eq. (1.138), Eq. (1.141) can be reduced to the form

$$\dot{x}'(t_*)R_3x(t_*) + x'(t_*)R_3\dot{x}(t_*) = 0. \tag{1.142}$$

For the one-dimensional case the moment t_* is determined by the following obvious equality

$$\dot{x}(t_*) = 0. \tag{1.143}$$

It was shown in [124] that the solution of the problem (1.140) exists if there exists a solution of the problem (1.137). Conditions of the kind (1.143) were studied also in [125].

A mean-square functional is one of the methods of account for control costs and estimation of the admissible power of control excitations. In many practical applications, direct constraints exist for the control excitation level. The most frequent form of constraints is $|u_j| \le U_0$, where u_j is the *j*-th component of the *m*-dimensional control vector. Thus, if the excitation is generated by the source of independent energy, then U_0 is the value of net tension; if the controlling element is a hydraulic or pneumatic drive, then pressure of water or gas, constrained by the drive directions (piston square, etc.), serves as a control.

We will restrict our examination to the one-dimensional system (1.110). The periodic regime equation and functional of the problem have the form

$$x(t) = \int_0^T \left[\chi_f(t-s)f(s) + \chi_u(t-s)u(s)\right]ds, \tag{1.144}$$

$$\Phi(u) = \frac{1}{T}\int_0^T x^2(s)ds \tag{1.145}$$

for the constraint $|u| \le U_0$. The Lagrange function of the problem (1.145), (1.144) has the form

$$L(s,x,u,p) = -\frac{1}{2}x^2(s) + p(s)x(s) - \int_0^T p(t)\left[\chi_f(t-s)f(s) + \chi_u(t-s)u(s)\right]dt,$$

stationary conditions with respect to x and maximum for u give

$$p(s) = x(s), \quad u(s) = -U_0 \operatorname{sgn}\int_0^T p(t)\chi_u(t-s)dt, \tag{1.146}$$

$$u(s) = -U_0 \operatorname{sgn}\int_0^T \chi_u(t-s)x(t)dt.$$

Thus, the control is a piecewise continuous function

$$u(t) = \begin{cases} -U_0, \, y(t) > 0, \\ U_0, \, y(t) < 0, \end{cases} \tag{1.147}$$

where

$$y(s) = \int_0^T \chi_u(t-s)x(t)dt, \tag{1.148}$$

and the problem is reduced to the search of switch points. The form of the function (1.147) shows the control realization method by means of elements with relay characteristic.

If the problem is the minimization of the function

$$\Phi(u) = x^2(t_*), \tag{1.149}$$

for the trajectories of the system (1.144) in the fixed point $t_* \in (0,T)$, then it is not complicated to obtain an optimal control relation

$$u(t) = -U_0 \operatorname{sgn}[\chi_u(t_* - t)x(t_*)]. \tag{1.150}$$

In particular, for the minimization problem of the functional (1.149) in the boundary point for $t_* = T - 0$, we have

$$u(t) = -U_0 \operatorname{sgn}[\chi_u(T - t)x(T)]. \tag{1.151}$$

1.5 Problems of Optimal Displacement for Linear Systems

In Section 1.4 the systems were considered, the movement of which was formed under the external excitation, and the control aim was the movement correction in accordance with certain quality criteria. Such type of corrections in applications usually means a feedback control.

But there is also another typical problem formulation for modern mechanisms: The movement is formed by means of controls, which are synthesized according to certain laws, while the external excitations are treated as disturbances. A programmed control formulated as a time function is used for construction of programmed movements.

The programmed control is used in various transport mechanisms, including manipulators. The control which is used for transportation of the system from one position to another is called a *position control.*

Below we limit our considerations to a single case of position control problems. Suppose that a movement program provides a return of the system into initial state and a multiple repetition of such cycles. Such type of movement is typical for manipulators in a conveyor system. (Different formulations for the problems of optimal position control of manipulators are analyzed, for instance, in [1, 2, 7, 17, 18, 161]).

Obviously, such movement can be analyzed as periodic, and each cycle – as the movement during one cycle. The cycle period is usually not given and is determined by demands of a high–speed action for fixed initial and final positions of an actuator. Positions of intermediate elements in the beginning and end of the period should be coordinated with the position of the actuator.

1.5.1 Systems with Symmetric Limiters

Let us examine one of the problems of the cyclic movement. Suppose that the movement of the points of the transport system of interest are described by the equations

$$x_j = l^j(p)u, \quad j = 1, 2, \ldots, m, \tag{1.152}$$

where $l^j(p) = m^j(p)/D(p)$ are operators of dynamic compliance of the elements, reduced to the point of application of the single control excitation $u(t)$; $x_j(t)$ are displacements of the system's points.

Suppose that the system moves from $x_j = -\Delta_j$ into position $x_j = \Delta_j$ and back with the speed damping in initial and final moments; $\dot{x}_j = v_j = 0$ for $x_j = \pm\Delta_j$. This is a simplified formulation for a problem of the movement of a multi-element manipulator, as far as usually the system stops after reaching the end points; the stop duration is determined by technology reasons. Furthermore, loading and unloading of the system occur in the positions $x_j = \pm\Delta_j$, i.e., its dynamic characteristics differ for direct and reverse movements. Excluding the stop duration from consideration, neglecting the change in dynamic characteristics and accounting for the asymmetric character of the movement, we can write the movement equation

$$x_j(t) \int_0^{T/2} \chi_2^j(t-s)u(s)ds, \quad 0 < t < T/2. \tag{1.153}$$

Here $\chi_2^j(t)$ is the periodic Green's function of the second kind (see Section 1.2). Eq. (1.153) describes the system movement over the first time interval, $0 < T < T/2$, and can be analytically expanded to the second interval, $T/2 < T < T$, accounting for conditions $x(t + T/2) = -x(t)$, $u(t + T/2) = -u(t)$. Then the functional for the high–speed action can be presented in the form

$$\Phi(u) = \int_0^{T/2} dt = \min. \tag{1.154}$$

The boundary conditions, fixing the beginning and of the interval, have the form

$$x_j(T/2) = -x_j(0) = \int_0^{T/2} \chi_2^j(T/2 - s)u(s)ds = \Delta_j, \tag{1.155}$$

$$v_j(T/2) = -v_j(0) = -\int_0^{T/2} \chi_{2s}^j(T/2-s)u(s)ds = 0 .$$

Let us construct the control $u(t)$, $|u| \le U$, which transforms the system from initial state into the final one in the shortest time. Including isoperimeter constraints (1.155) into the problem's functional, we get

$$\Phi(u) = \int_0^{T/2} \left\{ 1 + \sum_{j=1}^{m} \left[\lambda^j \chi_2^j(T/2-s) + \mu^j \chi_{2s}^j(T/2-s) \right] u(s) \right\} ds . \tag{1.156}$$

The problem is reduced to a construction of the control $u(s)$, minimizing the functional (1.156) for the condition (1.155). Here, the movement equations are not included into the number of problem's constraints.

According to the maximum principle, we get

$$u(s) = U_0 \operatorname{sgn} \left\{ \sum_{j=1}^{m} \left[\lambda^j \chi_2^j(T/2-s) + \mu^j \chi_{2s}^j(T/2-s) \right] \right\} . \tag{1.157}$$

Eqs. (1.54) are used together with condition (1.79) for determination of coefficients μ^j, λ^j which fix switch points. Eq. (1.57) can be considerably simplified, if the constraints are applied only to the actuator movement $(j=1)$, and initial positions and velocities of other elements are determined by optimality conditions and Eq. (1.153). Then

$$u = U_0 \operatorname{sgn}\left[\lambda^1 \chi_2^1(T/2-s) + \mu^1 \chi_{2s}^1(T/2-s) \right] . \tag{1.158}$$

Substituting (1.158) into (1.155), we get

$$v_1(T/2) = -U_0 \int_0^{T/2} \chi_{2s}^1(T/2-s) \operatorname{sgn}\left[\lambda^1 \chi_2^1(T/2-s) + \mu^1 \chi_{2s}^1(T/2-s) \right] ds = 0 ,$$

from which $\mu^1 = 0$ and

$$u(s) = U_0 \operatorname{sgn} \lambda^1 \operatorname{sgn} \chi_2^1(T/2-s) \tag{1.159}$$

follow. Thus,

$$x_1(T/2) = U_0 \int_0^{T/2} \left| \chi_2^1(T/2-s) \right| ds \operatorname{sgn} \lambda^1 = \Delta_1 > 0 . \tag{1.160}$$

It follows from the last relation, that $\lambda_1 > 0$, and the period T is determined by the equation

$$U_0 \int_0^{T/2} \left| \chi_2^1(t) \right| dt = \Delta . \tag{1.161}$$

Eq. (1.79) can be used for calculation of λ^1. Coordinates and velocities of other points at the beginning and end of the process are determined by relations

$$x_j(T/2)=U_0\int_0^{T/2}\chi_2^j(T/2-s)\,\mathrm{sgn}\,\chi_2^1(T/2-s)ds\,,$$
$$x_j(0)=-x_j(T/2) \tag{1.162}$$

$$v_j(T/2)=U_0\int_0^{T/2}\chi_{2s}^j(T/2-s)\,\mathrm{sgn}\,\chi_2^1(T/2-s)ds\,,$$
$$v_j(0)=-v_j(T/2) \tag{1.163}$$

Let us analyze an example of control for the simplest positioning system (Fig. 1.8). The control excitation $u_2(t)$, $|u_2|\le U$, is applied to the element *2* which is connected by the elastic link with the actuator *1*. Considering each element as a solid and neglecting properties of the motor, we can write the motion equations of the system in the form

$$m_2\ddot{x}_2+c_2x_2+c_1(x_2-x_1)=u_2\,,$$
$$m_1\ddot{x}_1+c_1(x_1-x_2)=0\,. \tag{1.164}$$

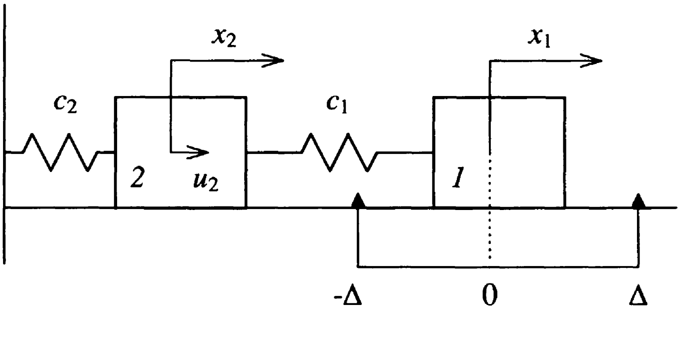


Fig. 1. 8

Here m_1, m_2 are masses of elements, c_1, c_2 are stiffness of elastic links, x_1, x_2 are coordinates of mass centers of the elements with respect to the equilibrium state. Suppose that the actuator has a reversible movement between two symmetric limiters with coordinates $x_1=\pm\Delta$ and with zero velocity at moments of entering and leaving the limiters: $\dot{x}_1 = 0$ for $x_1 = \pm\Delta$.

Excluding the variable x_2 from Eqs. (1.164), we get

$$x_1 = l^1(p)u\,, \quad l^1(p) = \gamma_1^2 \Delta^{-1}(p)\,,$$

$$\Delta(p) = (p^2 + \gamma_1^2)(p^2 + \gamma_1^2 + \gamma_2^2) - \gamma^2\gamma_1^2\,, \tag{1.165}$$

$$\gamma_1^2 = c_1/m_1\,, \quad \gamma_2^2 = c_2/m_2\,, \quad \gamma^2 = c_2/m_1\,, \quad u = u_2/m_2\,.$$

Respectively, according to (1.54)

$$\chi_2^1(t) = \frac{\gamma_1^2}{\Omega_2^2 - \Omega_1^2}\left[\frac{1}{2\Omega_1}\frac{\sin\Omega_1(t - T/4)}{\cos\Omega_1 T/4} - \frac{1}{2\Omega_2}\frac{\sin\Omega_2(t - T/4)}{\cos\Omega_2 T/4}\right], \tag{1.166}$$

where $p = \pm i\Omega_1$, $\pm i\Omega_2$ are roots of characteristic equation $\Delta(p) = 0$, $\Omega_1 < \Omega_2$. Obviously, the movement frequency ω in the system, optimal with respect to the high–speed action, should be higher than the maximum eigenfrequency Ω_2:

$$\omega > \Omega_2\,, \; T = 2\pi/\omega < 2\pi/\Omega_2\,. \tag{1.167}$$

The lower the motor capacity (U_2) is, the closer is the movement to the resonance with a frequency, insignificantly higher than Ω_2.

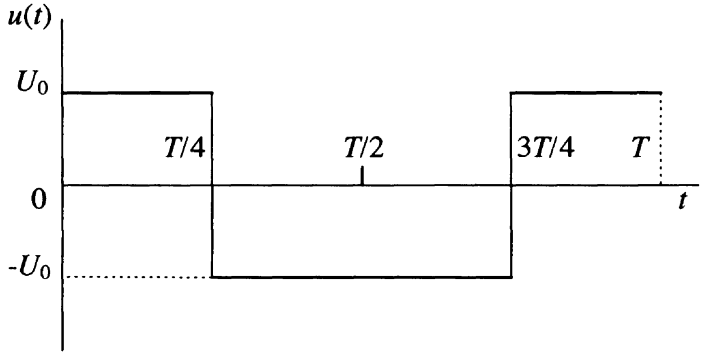


Fig. 1.9

Analogous statements hold for conservative oscillatory systems with an arbitrary number of degrees of freedom. Consider $\Omega_{1,2} < \omega$. Suppose that the function $\chi_2^1(T/2 - t) = -\chi_2^1(-t)$ has only one root $t = T/4$, and optimal control (Fig. 1.9) has only one switch point

$$u(t)=\begin{cases}-U_0, 0<t<\dfrac{T}{4} \\ U_0, \dfrac{T}{4}<t<\dfrac{T}{2}, U_0=\dfrac{U_2}{m_2}\end{cases} \tag{1.168}$$

Substituting (1.166) into (1.161), we get the equation for determination of the period T

$$\frac{\gamma_1^2 U_0}{\Omega_2^2-\Omega_1^2}\left[\frac{1-\cos\Omega_1 T/4}{\Omega_1^2\cos\Omega_1 T/4}-\frac{1-\cos\Omega_2 T/4}{\Omega_2^2\cos\Omega_2 T/4}\right]=\Delta\,. \tag{1.169}$$

After determination of the period T, it is necessary to prove the condition (1.167) and to ascertain the sufficiency of the motor capacity for the regime realization with the desired period.

The initial and final positions of the actuator *1* are determined by the contact conditions, and there are no constraints for the guiding element 2. At the same time, the initial position and velocity of the element 2 should be chosen according to periodicity conditions and be coordinated with the movement of the element *1* in order to realize the T-periodic regime in the system.

According to (1.164), the movement of the guiding element 2 is described by the equation

$$x_2=l^2(p)u\,,\quad l^2(p)=\Delta^{-1}(p)\left(p^2+\gamma_1^2\right)u\,, \tag{1.170}$$

where $\Delta(p)$ is the same expression as in (1.165). Then

$$\chi_2^2(t)=\frac{1}{\Omega_2^2-\Omega_1^2}\left[\frac{\gamma_1^2-\Omega_1^2}{2\Omega_1}\frac{\sin\Omega_1(t-T/4)}{\cos\Omega_1 T/4}-\frac{\gamma_1^2-\Omega_2^2}{2\Omega_2^2}\frac{\sin\Omega_2(t-T/4)}{\cos\Omega_2 T/4}\right]. \tag{1.171}$$

Substituting (1.171) into (1.163), we get

$$v_2(0)=-v_2(T/2)=0\,,\quad x_2(0)=-x_2(T/2)\,, \tag{1.172}$$

$$x_2(T/2)=\frac{U_0}{\Omega_2^2-\Omega_1^2}\left[\frac{\gamma_1^2-\Omega_1^2}{\Omega_1^2}\frac{1-\cos\Omega_1 T/4}{\cos\Omega_1 T/4}-\frac{\gamma_1^2-\Omega_2^2}{\Omega_2^2}\frac{1-\cos\Omega_2 T/4}{\cos\Omega_2 T/4}\right].$$

Conditions (1.172) determine the initial and final positions of the element 2 necessary for the realization of the optimal T-periodic movement.

1.5.2
Systems with Asymmetric Characteristics

Up to now, the problem of cyclic displacement with return of the system to initial state at the end of the process was analyzed. But the obtained results are true also for a case of the displacement only in a positive direction for $0 < t < T/2$, with damping of oscillations in the extreme points. It means that the displacement from the initial point into the final one can be interpreted as the first interval (half-period) of asymmetric periodic movement. Such a "supplement" of the motion to periodic one allows us to reduce the optimal displacement problems to the periodic control problems. Thus, the control (1.168) in the studied example perform the displacement of the actuator from $x^1 = -\Delta$ into the position $x^1 = \Delta$ in the minimum time $T/2$; the interval $t = T/2$ is determined by Eq. (1.169).

Such an approach expands the set of optimal displacement problems, which can be treated by methods of periodic optimization. Consider a problem of asymmetric cyclic displacement. Suppose that actuator is loaded at the moment when it reaches the right limiter; after that the system returns in initial state, where unloading occurs. Thus, dynamic characteristics of the system for direct and inverse movement are different, though the entire process remains periodic.

Consider at first a movement of a one-dimensional system. Let $l^{\pm}(p)$ be operators of dynamic compliance, linking the control and movement of a control point in positive and negative directions, and

$$\chi_2^{\pm}(t) = \frac{2}{T}\sum_{k=-\infty}^{\infty} l^{\pm}[(2k+1)i\omega]\exp[(2k+1)i\omega t]$$

be respective Green's functions of the second kind.

We will divide the movement interval into two parts and consider the movement in the positive direction as the first half-period of the periodic movement of the system with the dynamic compliance operator $l^{+}(p)$ and the movement in the negative direction – as the first half-period of the system movement with the dynamic compliance operator $l^{-}(p)$.

For each part, dynamics of the system is described by the equation

$$x^{\pm}(t) = \int_0^{T^{\pm}/2} \chi_2^{\pm}(t-s)u^{\pm}(s)ds, \tag{1.173}$$

with boundary conditions

$$\begin{aligned} &x^{+}(0) = x^{-}(T^{-}/2) = -\Delta, \quad x^{+}(T^{+}/2) = x^{-}(0) = \Delta, \\ &v^{+}(0) = v^{-}(T^{-}/2) = 0, \quad v^{+}(T^{+}/2) = v^{-}(0) = 0. \end{aligned} \tag{1.174}$$

The total movement duration is $T = (T^{+} - T^{-})/2$. For each of this parts, the control $u^{\pm}(t)$, $|u^{\pm}| \le U_0$ should be found, which performs the displacement in the shortest time. The optimal control for the movement in the positive direction is determined by Eq. (1.159), and the value of $T^{+}/2$ – by Eq. (1.161):

$$u^+(t)=U_0 \operatorname{sgn}\chi_2^+(T^+/2-t),\qquad U_0\int_0^{T^+/2}\left|\chi_2^+(t)\right|dt=\Delta. \tag{1.175}$$

The movement in a negative direction has an opposite sign, i.e.,

$$u^-(t)=-U_0 \operatorname{sgn}\chi_2^-(T^-/2-t),\qquad U_0\int_0^{T^-/2}\left|\chi_2^-(t)\right|dt=\Delta. \tag{1.176}$$

Eqs. (1.175) serve as continuity conditions of the coordinate x and velocity v of the actuator at transition from one trajectory to another. If the system has more than one degree of freedom, then generalized coordinates describing movements of input and intermediate elements should also satisfy conditions of continuity and periodicity.

Let $y^{\pm}(t)$ be a one-dimensional vector of generalized coordinates of input and intermediate elements of the system and $K_2^{\pm}(t)$ be matrix periodic Green's functions, corresponding to the movements for each interval

$$y^{\pm}(t)=\int_0^{T^{\pm}/2}K_2^{\pm}(t-s)u^{\pm}(s)ds, \tag{1.177}$$

where $u^{\pm}(s)$ is a scalar control to be found.

If a switch of dynamic characteristics occurs instantly, then the continuity condition

$$y^+(T^+/2)=y^-(0),\qquad y^-(T^-/2)=y^+(0) \tag{1.178}$$

should be included into constraints of the problem, and the control u^+, u^- should be constructed with account for (1.178). The structure of the control is thus considerably complicated.

If the switch is preceded by a stop as it is a case in real technical systems, then another approach is expedient. The control should be chosen according to (1.173) – (1.176), and intermediate elements should be transferred into the initial point of a new trajectory during the stop. It means, that during the stop at the right limiter, $x^+(T^+/2)=\Delta$, the intermediate elements are transferred from the state $y^+(T^+/2)$ into the state $y^-(0)\neq y^+(T^+/2)$. In the same way, during the stop at the left limiter $x^-(T^-/2)=-\Delta$, the intermediate elements are transferred from the state $y^-(T^-/2)$ into the state $y^+(0)\neq y^-(T^-/2)$. Thus, at the start of each half-interval all the points of the system will be in the state, corresponding to the initial conditions of the asymmetric periodic movement to be found. Such an approach is also possible in the case when the actuator makes not simply one but several stops with change of dynamic characteristics of the system during the cycle.

Consider once more the mechanism shown in Figure 1.8. Suppose that the loading takes place in the point $x_1 = \Delta$, and unloading – in the point $x_1 = -\Delta$, i.e., the mass of the first element is equal to m_1 for the movement in the positive direction, and for the movement in the negative direction is equal to $m > m_1$. The system parameters change respectively: for the movement in the positive direction $\Omega_1^+ = \Omega_1$, $\Omega_2^+ = \Omega_2$, $\gamma_1^+ = \gamma_1$, for the movement in the negative direction $\Omega_1^- < \Omega_1$, $\Omega_2^- < \Omega_2$, $\gamma_1^- < \gamma_1$.

Thus, the system movement in the positive direction is determined by Eqs. (1.165) – (1.173). For the movement in the negative direction, the control structure (1.168) remains but the interval is $\Delta t^- = T^-/2 \neq T^+/2$. The value of T^- can be obtained from Eq. (1.169) with the substitution $\gamma_1 \to \gamma_1^-$, $\Omega_1 \to \Omega_1^-$, $\Omega_2 \to \Omega_2^-$. It is obvious that $T^- > T^+ = T$. The movement of the first element is continuous:

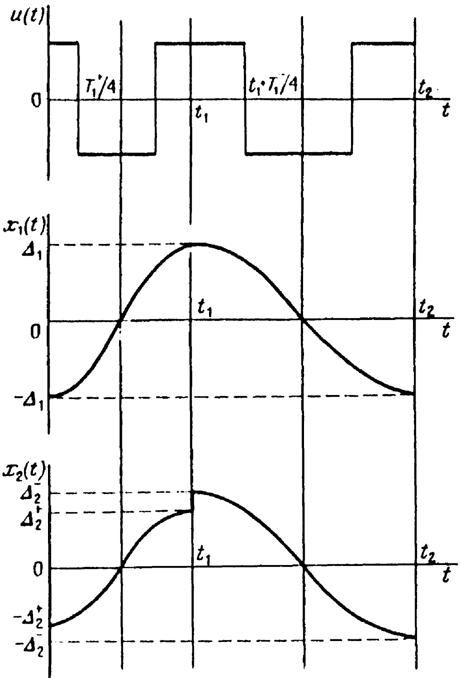


Fig. 1.10

$$x_1^+(T^+/2) = x_1^-(0) = \Delta,$$
$$x_1^-(T^-/2) = x_1^+(0) = -\Delta,$$
$$v_1^+(T^+/2) = v_1^-(0) = 0,$$
$$v_1^-(T^-/2) = v_1^+(0) = 0.$$

For the second element the following relations hold:

$$v_2^+(T^+/2) = v_2^-(0) = 0,$$
$$v_2^-(T^-/2) = v_2^+(0) = 0,$$

but $x_1^+(T^+/2) \neq x_1^-(0)$,

$$x_2^-(T^-/2) \neq x_2^-(0).$$

The coordinate $\Delta_2^+ = x_2^+(T^+/2)$ is determined by Eq (1.172), the coordinate $\Delta_2^- = x_2^-(0) = -x_2^-(T^-/2)$ is determined by Eq. (1.162) for $T \to T^-$, $\Omega_1 \to \Omega_1^-$, $\Omega_2 \to \Omega_2^-$. Hence, during the technological stop of the actuator at the right limi, at $t = T^+/2$, the element *2* should be transferred from the position Δ_2^+ in the position Δ_2^-. The respective transition is carried out also during the stop at the left limiter, at $t = T^{-}/2$. Graphs of the functions $x_1(t)$, $x_2(t)$ and of the control $u(t)$ are presented in Figure 1. 10.

1.5.3 Systems with Asymmetric Limiters

Up to now, it was considered that the limiters fixing initial and final positions of the actuator are symmetrically situated. Owing to symmetry, the following equations hold:

$$x(t+T/2) = -x(t)\,, \quad u(t+T/2) = -u(t)\,,$$

so the movement can be examined only at one half-period. Consider a more complicated case of the movement between tow asymmetrically situated limiters with stops at each of them.

Consider only the case of the one-dimensional system. Let the movement of the actuator be described by the equation

$$x = l(p)u, \tag{1.179}$$

where x is the displacement of the center of mass of the actuator, u is a guiding excitation, $l(p)$ is a respective operator of dynamic compliance. Suppose that the system moves from the initial point to the final one with stops at the beginning and end of the process

$$x = -\Delta_1\,, \quad v = 0\,, \quad x = -\Delta_2\,, \quad v = 0\,. \tag{1.180}$$

In spite of the asymmetry of the limiters, the process has a periodic character, and each cycle can be treated as a movement during one period.

As above, we will find the control $u(t)$, $|u| \le U_0$, which realizes the cycle in the shortest time. Introduce a new variable

$$\tilde{x} = x + \delta\,, \quad \delta = (\Delta_1 - \Delta_2)/2\,. \tag{1.181}$$

Then the control is reduced to the form

$$\tilde{x} = l(p)u - \delta\,, \tag{1.182}$$

and the boundary conditions become symmetric

$$\begin{aligned} &\tilde{x} = -(\Delta_1 + \Delta_2)/2\,, \quad \dot{\tilde{x}} = 0\,, \\ &\tilde{x} = (\Delta_1 + \Delta_2)/2\,, \quad \dot{\tilde{x}} = 0\,. \end{aligned} \tag{1.183}$$

Obviously, the asymmetric control $u(t+T/2) = -u(t)$ does not satisfy Eq. (1.182), as far as there is a constant component in the right–hand part of it. Introduce the function

$$\tilde{u} = u - u_0, \tag{1.184}$$

where $u_0 = \delta l^{-1}(0)$. Accounting for (1.184), Eq. (1.182) is reduced to the form

$$\tilde{x} = l(p)\tilde{u}. \tag{1.185}$$

Eq. (1.185) with boundary conditions (1.183) has an asymmetric solution

$$\tilde{x}(t) = \int_0^{T/2} \chi_2(t-s)\tilde{u}(s)ds, \quad 0 < t < T/2, \tag{1.186}$$
$$\tilde{x}(t+T/2) = -\tilde{x}(t),$$

where $\chi_2(t)$ is still the periodic Green's function of the second kind for the system (1.185).

If control $u(t)$ satisfies conditions $-U_0 \le u \le U_0$, then for the function $\tilde{u}(t)$

$$-U_0 - u_0 \le \tilde{u} \le U_0 - u_0 \tag{1.187}$$

is true. Thus, the problem is reduced to the minimization of the functional of the high-speed action (1.154) at trajectories (1.186) with constraints (1.183), (1.187).

Working as in Section 1.51, we get

$$\tilde{u}(s) = \begin{cases} U_+, & f(s) > 0, \\ U_-, & f(s) < 0, \end{cases} \tag{1.188}$$

where

$$f(s) = \lambda\chi_2(T/2 - s) + \mu\chi_{2s}(T/2 - s)$$
$$U_+ = U_0 - u_0, \quad U_- = -U_0 - u_0. \tag{1.189}$$

The boundary conditions (1.183) can be used for determination of switch moments and of the optimal cycle duration.

Let us construct a regime with one switch point. Let $u_0 < U_0$. Then

$$\tilde{u}(s) = U_+ > 0, \quad 0 < s < t_*,$$
$$\tilde{u}(s) = U_- < 0, \quad t_* < s < T/2. \tag{1.190}$$

From (1.183), (1.190), we get

$$\tilde{x}(T/2) = U_+\int_0^{t_*} \chi_2(T/2 - s)ds + U_-\int_{t_*}^{T/2} \chi_2(T/2 - s)ds, \tag{1.191}$$

$$v(T/2) = -U_+\int_0^{t_*} \chi_{2s}(T/2 - s)ds - U_-\int_{t_*}^{T/2} \chi_{2s}(T/2 - s)ds =$$
$$= -\chi_2(T/2 - t_*)(U_+ - U_-) + \chi_2(T/2)(U_+ - U_-).$$

The last equality serves for determination of the switch point. According to (1.189), we get the equation

$$\chi_2(T/2-t_*)=-\frac{u_0}{U_0}\chi_2(T/2). \tag{1.192}$$

In particular, for the system with one degree of freedom for

$$\chi_2=\frac{\sin\Omega(t-T/4)}{2\cos\Omega T/4}$$

we get

$$\sin\Omega(t_*-T/4)=\frac{u_0}{U_0}\sin\Omega T/4. \tag{1.193}$$

1.6 Periodic Control for Quasi-linear Systems

If a system contains non-linear or non-conservative links, then it is, as a rule, impossible to construct an analytical solution, and the determination of the optimal control needs not only the formal use of the maximum principle but also the utilization of approximate methods.

We will give some schemes of successive approximations which allow us to obtain the optimal control with desired degree of accuracy. We will use the known results of the application of the method of a small parameter for construction of periodic solutions for systems of differential equations [84, 85, 100].

1.6.1 Periodic Control in Systems Described by Differential Equations

Suppose that dynamics of the system is described by the equation

$$\dot{x}=Ax+Bu+\varepsilon f(t,x)+F(t), \tag{1.194}$$

where $x\in R_n$, $u\in R_m$, A, B are matrixes of corresponding dimension, f and F are T-periodic in t vector functions, ε is a small parameter. The systems is considered to be non-resonant, i.e., eigenvalues of the matrix A differ from $\pm i2\pi k/T$ ($k=0$, ±1,...). The requirements to the functions f and F should be discussed below.

Let us find a T-periodic control, minimizing the functional

$$\Phi_\varepsilon(u)=\int_0^T[\varphi(t,x)+\psi(u)]dt \tag{1.195}$$

on the T-periodic solution $x(t)$ of the system (1.194) under condition $u\in U\subset R_m$, where U is compact in R_m. The function φ is periodic in T.

The optimal control u_* is determined from the condition (1.86)

$$u_*(t) = \arg\max_{u \in U}\left[q'\left(Ax + Bu + \varepsilon f(t,x) + F(t)\right) - \varphi(t,x) - \psi(u)\right], \tag{1.196}$$

where $q(t)$ is the T-periodic solution of the system (1.85)

$$\dot{q} = -A'q - \varepsilon f_x'(t,x)q + \varphi_x(t,x). \tag{1.197}$$

From (1.196) we have

$$u_*(t) = V(q(t)). \tag{1.198}$$

Thus, the solution of the periodic optimization problem is reduced to the periodic solution construction for the non-linear system which was studied in [84]:

$$\begin{aligned} \dot{x} &= Ax + BV(q) + F(t) + \varepsilon f(t,x), \\ \dot{q} &= -A'q + \varphi_x'(t,x) - \varepsilon f_x'(t,x)q. \end{aligned} \tag{1.199}$$

Consider that right parts of Eqs. (1.199) satisfy the following condition.

Theorem 1.3. *Let*

1) *the functions f(t,x), φ(t,x) be uniquely determined for* $t \in (-\infty,\infty)$, $x \in R_n$, *continuous and T-periodic in t uniformly for all* $x \in R_n$, *three times differentiable with respect to x for* $x \in R_n$ *uniformly for* $t \in (-\infty,\infty)$; *the function F(t) is bounded and T-periodic in t;*

2) *the function V(q) is uniquely determined over domain G of the change of g;*

3) *the domain G can be divided by surfaces*

$$v_k(q) = 0 \tag{1.200}$$

into domains $G_1, G_2, \ldots, G_l$, *in each of which the function V is continuous and two times differentiable till the boundaries; on the surfaces* (1.200) *there are discontinuities of the first kind of the function V or of the derivatives* V_q, V_{qq}; *functions* $v_k(q)$ *are two times continuously differentiable with respect to q for* $q \in G$;

4) *the generating system*

$$\dot{x}^0 = Ax^0 + BV(q) + F(t), \quad \dot{q}^0 = -A'q^0 + \varphi_x(t,x^0) \tag{1.201}$$

has a unique isolated T-periodic solution (x^0, q^0), *for which conditions*

$$\left.\frac{dv_k}{dq}, \frac{dq}{dt}\right|_{q=q^0} \neq 0 \tag{1.202}$$

hold in crossing points with the surfaces (1.200) (conditions for existence of the unique isolated solution are given in [84, 100]).

Then there exists the unique periodic solution (x_*, q_*) *of the disturbed system* (1.199) *in the* ε*-zone of the generated solution, and the scheme of the successive approximations holds* [84]

$$\begin{aligned} &\dot{x}^l = Ax^l + BV(q^l) + F(t) + \varepsilon f(t, x^{l-1}), \\ &\dot{q}^l = -A'q^l + \varphi_x(t, x^l) - \varepsilon f_x'(t, x^{l-1})q^{l-1}, \\ &x^l(t+T) = x^l(t), \quad q^l(t+T) = q^l(t), \end{aligned} \tag{1.203}$$

and

$$\left|x^l - x_*\right| \le c\varepsilon^{l+1}, \quad \left|q^l - q_*\right| \le c\varepsilon^{l+1}. \tag{1.204}$$

Here and below c is a constant independent of ε.

1.6.2
The Method of Successive Approximations for Integral Equations of Periodic Movement

The integral equation of the periodic movement of the system (1.194) has the form

$$x(t) = \int_0^T K(t-s)\left[Bu(s) + F(s) + \varepsilon f(s, x)\right]ds, \tag{1.205}$$

where

$$K(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}(ki\omega I - A)^{-1}\exp(ki\omega t) \tag{1.206}$$

is a matrix periodic Green's function of the linear part of the system (1.194).

Let us write the Lagrange function of the problem (1.205), (1.195)

$$L(s, x, u, P) = P'(s)x(s) - Q_0(s, x, u, P) - \varepsilon Q_1(s, x, P), \tag{1.207}$$

where

$$Q_0(s, x, u, P) = \varphi(s, x(s)) + \psi(u(s)) + \int_0^T P'(t)K(t-s)dt\left[Bu(s) + F(s)\right], \tag{1.208}$$

$$Q_1(s, x, P) = \int_0^T P'(t)K(t-s)dt f(s, x),$$

and the Lagrange multiplier satisfies the equation $\partial L/\partial x = 0$, or

$$P(s) = \varphi_x(s, x) + \varepsilon f_x(s, x)\int_0^T K'(t-s)P(t)dt. \tag{1.209}$$

The control u_* is determined by the maximum condition

$$u_*(t) = \arg\max_{u \in U} L(s, x, u, P) = \arg\max_{u \in U}\left[-\psi(u) + q'Bu\right] = V\left(q(s)\right), \tag{1.210}$$

where

$$q(s) = -\int_0^T K'(t-s)P(t)dt. \tag{1.211}$$

Owing to (1.206), (1.211), the variables $q(s)$, $P(s)$ can be linked by the relation

$$\dot{q} + A'q = P. \tag{1.212}$$

From (1.209), (1.212) follows, that the function $q(s)$ coincides with the periodic solution of Eq. (1.197), and the function (1.198) – with Eq. (1.200) [compare (1.91), (1.94)].

Let us write the following scheme of the successive approximations:

$$\begin{aligned}
&x^0(t) = \int_0^T K(t-s)\left[Bu^0(s) + F(s)\right]ds, \\
&u^0(s) = V\left(q^0(s)\right), \\
&q^0(s) = -\int_0^T K'(t-s)P^0(t)dt, \\
&P^0(s) = \varphi_x\left(s, x^0(s)\right);
\end{aligned} \tag{1.213}$$

$$\begin{aligned}
&x^l(t) = \int_0^T K(t-s)\left[Bu^l(s) + F(s) + \varepsilon f\left(s, x^{l-1}(s)\right)\right]ds, \\
&u^l(s) = V\left(q^l(s)\right), \\
&q^l(s) = -\int_0^T K'(t-s)P^l(t)dt, \\
&P^l(s) = \varphi_x\left(s, x^l(s)\right) + \varepsilon f_x\left(s, x^{l-1}(s)\right)q^{l-1}(s).
\end{aligned} \tag{1.214}$$

It is easy to see, that the scheme (1.213), (1.214) is an integral form of the equations of periodic movement (1.199), (1.203), and for the approximations (1.214) the estimates (1.205) hold.

We will show next that the control $u^l(s)$ is quasi-optimal with respect to the initial system, i.e.,

$$\Phi_\varepsilon\left(u^l\right) - \Phi_\varepsilon\left(u_*\right) \le c\varepsilon^{l+1}, \tag{1.215}$$

where $\Phi_\varepsilon\left(u^l\right)$ is the value of the functional (1.195) for trajectories of the system

(1.194) [or (1.205)] for $u = u^l$. With this purpose we will prove that

$$\left|\int_0^T \left[u_*(s) - u^l(s)\right]ds\right| \le c\varepsilon^{l+1},$$

$$\left|x_*(s) - x_*^l(s)\right| \le c\varepsilon^{l+1}, \tag{1.216}$$

where $x_*^l(s)$ is the solution of the system (1.195) for $u = u^l(s)$.

Let us limit our considerations for the sake of obviousness by a scalar case: the function $V(q)$ is a scalar depending on one component q_j of the vector q [compare (1.96) – (1.100)], and there exists the unique discontinuity line $v(q_j) = 0$ (the index j is omitted below). Let $\eta^l \varepsilon^{l+1} = q^l - q_*$: owing to (1.204) $\eta^l = O(1)$. Then

$$\varepsilon^{-(l+1)}\left[u_*(s) - u^l(s)\right] = \varepsilon^{-(l+1)}\left[V\left(q_* + \varepsilon^{l+1}\eta^l\right) - V(q_*)\right]. \tag{1.217}$$

For $\varepsilon \to 0$ the last expression can be treated as a generalized derivative [70], i.e.,

$$\varepsilon^{-(l+1)}\int_0^T \left[u_*(s) - u^l(s)\right]ds = \int_0^T \left[\sum_{k=1}^m U_k \delta\left(q_*(s) - a_k\right)\eta^l(s) + \left\{V_q\left[q_*(s)\right]\right\}\eta^l(s)\right]ds +$$
$$+O(\varepsilon) = I_1 + I_2 + O(\varepsilon), \tag{1.218}$$

where $\{V(q)\}$ is the "smooth" part of the function $V(q)$, $\delta(q)$ is the Dirac delta function, a_k are the roots of the equation $v(q_i) = 0$, $U_k = V(a_k + 0) - V(a_k - 0)$ is the discontinuity value. Using the properties of δ-function and accounting for (1.202), (1.204), let us write

$$I_1 = \sum_{k=1}^m \sum_{\nu=1}^n U_k \left|\left[\dot q^0(s_k^\nu)\right]^{-1}\right| \eta^l(s_k^\nu) + O(\varepsilon), \tag{1.219}$$

where s_k^ν are the roots of the equation $q^0(s) = a_k$. It follows from the condition (1.202) that the roots a_k, s_k^ν are isolated, and transformations (1.218), (1.219) hold. Thus, the first item in (1.218) is bounded, the boundedness of the second item is obvious. Therefore, the first estimation (1.208) holds. Moreover, using analogous considerations, it is not complicated to demonstrate, that

$$I(t) = \left|\int_0^T \rho(t,s)\left[u_*(s) - u^l(s)\right]ds\right| \le c\varepsilon^{l+1}, \tag{1.220}$$

if the function $|\rho(t,s)| \le M$ for $0 \le t,\ s \le T$, and, at last,

$$\left|\int_0^T \psi\left[(u_*(s)) - \psi\left(u^l(s)\right)\right]ds\right| \le c\varepsilon^{l+1} \tag{1.221}$$

for any continuously differentiable function $\psi(u)$. The estimations (1.216)–(1.221) can be easily generalized also for the multidimensional case.

Next let us write

$$x_*(t) - x_*^l(t) = \int_0^T K(t-s)B\left[u_*(s) - u^l(s)\right]ds + \varepsilon\int_0^T K(t-s)\left[f(s,x_*) - f\left(s,x_*^l\right)\right]ds.$$

Accounting for (1.220) and for the first condition of Theorem 1.3, we get

$$\left|x_*(t) - x_*^l(t)\right| \le C\varepsilon^{l+1} + \varepsilon M\int_0^T \left|x_*(s) - x_*^l(s)\right|ds, \tag{1.222}$$

where C, M are bounded constants. Define

$$\int_0^T \left|x_*(s) - x_*^l(s)\right|ds = \mu.$$

Integrating both parts of (1.220) with respect to t over the interval $[0,T]$, we get

$$\mu \le \left[C\varepsilon^{l+1} + \varepsilon\mu M\right]T,$$

from which follows, that for sufficiently small ε

$$\mu \le C\varepsilon^{l+1}\left|1 - \varepsilon MT\right|^{-1},$$

i.e., according to (1.222)

$$\left|x_*(t) - x_*^l(t)\right| \le c\varepsilon^{l+1}. \tag{1.223}$$

At last, using (1.221), (1.223), we get the inequality (1.215).

Thus, the following theorem is true.

Theorem 1.4. *Let functions* $f(t,x)$, $\varphi(t,x)$, $V(q)$ *satisfy the conditions of* Theorem 1.3, *the function* $\psi(u)$ *is continuous together with its first derivative for* $u \in U \subset R_m$.

Then for sufficiently small ε, *the control* u^l *found in the l-th approximation* (2.214) *is quasi-optimal with respect to the disturbed system with estimations* (1.215), (1.216).

If the function $V(q)$ is continuous and satisfies the Lipschitz condition over the surfaces (1.200), then it follows from (1.217), that the estimate (1.216) can be strengthened:

$$\left|u_*(s) - u^l(s)\right| \le c\varepsilon^{l+1}. \tag{1.224}$$

The Theorems 1.3 and 1.4 reduce the construction of successive approximations to the approximated solution of the full system of equations of the maximum principle. We will show that the accuracy of approximations would not change, if we would search the successive approximations $\tilde{u}^l$ as minimizing at each step the functional (1.195) for the periodic solution of the system

$$\dot{x} = Ax + Bu + \varepsilon f^{l-1}(t,x) + F(t), \tag{1.225}$$

where $f^{l-1}(t) = f\left(t, \tilde{x}^{l-1}(t)\right)$ and $\tilde{x}^{l-1}(t)$ is the solution of the previous optimal problem (the analogous approach was developed in [81]). Here, at each stage the optimal control is found on the trajectories of the linear system and

$$\tilde{u}^l(t) = \arg\max_{u \in U}\left[q'\left(Ax + Bu + \varepsilon f^{-1}(t) + F(t)\right) - \varphi(t,x) - \psi(u)\right] = V(q(t)). \tag{1.226}$$

Here $V(q(t))$ is the same relation (1.198), $q(t)$ is a T-periodic solution of the following system

$$\dot{q} = -A'q + \varphi_x(t,x). \tag{1.227}$$

Obviously, the solution $\left(\tilde{x}^0, q^0\right)$ of the generating problem coincides with the one, found according to the schemes of Theorems 1.3 and 1.4, and there exists the unique solution (x_*, q_*) of the disturbed system (1.199), situated in the ε-zone of the generating solution.

Let us now examine the linear system

$$\dot{x} = Ax + Bu + F(t) + \varepsilon f_*(t,x) \tag{1.228}$$

instead of the system (1.194), where $f_*(t) = f_*\left(t, x_*(t)\right)$. Obviously, for $u_* = u_*(t)$, the solution of the system (1.228) coincides with $x_*(t)$. Moreover, owing to the uniqueness of the optimal problem solution, the control $\tilde{u}_*$ minimizing the functional (1.195) on the trajectories of the system (1.228) coincides with u_*. Thus, the problem (1.194), (1.195) is equivalent to the problem (1.195), (1.228). Then, as can be easily proved,

$$\tilde{u}_* = u_*(t) = V\left(q_*(t)\right),$$

where q_* is the solution of the problem (1.227), (1.228).

Let us prove the convergence of successive approximations $\left(\tilde{x}_l, \tilde{q}_l\right)$ to (x_*, q_*). We will limit our considerations, as above, to a one-dimensional case, considering that the function $V(q)$ is a scalar function which depends on one of the components

q_j of the vector q, the function $\varphi(x)$ d ds on the component x_1 of the vector x and all the components of the vector B $B_j = 1$ are equal to zero.

Then

$$\tilde{x}^1(t) = \int_0^T \chi_{1j}(t-s)\left[V\left(\tilde{q}^1(s)\right) + F(s) \cdot \quad (s)\right]ds,$$

(1.229)

$$\tilde{q}^1(t) = -\int_0^T \chi_{1j}(s-t)\varphi_x\left(\tilde{x}^1(s)\right)ds,$$

where $\chi_{1j}(t)$ is an element $(1,j)$ of the rix $K(t)$, indices by x_1, q_1 are omitted, $x = x_1$, $q = q_1$.

Respectively,

$$x_*(t) = \int_0^T \chi_{1j}(t-s)\left[V\left(q_*(s)\right) + F(s) + \quad s)\right]ds,$$

(1.230)

$$q_*(t) = -\int_0^T \chi_{1j}(s-t)\varphi_x\left(x_*(s)\right)ds.$$

From Theorem 1.3 follows, that

$$\left|\tilde{x}^1 - x^0\right| \le c\varepsilon, \quad \left|x_* - x^0\right| \le c\varepsilon, \qquad (1.231)$$
$$\left|\tilde{q}^1 - q^0\right| \le c\varepsilon, \quad \left|q_* - q^0\right| \le c\varepsilon,$$

i.e.,

$$\left|\tilde{x}^1 - x_*\right| \le c\varepsilon, \quad \left|\tilde{q}^1 - q_*\right| \le c\varepsilon$$

and functions

$$\xi^1 = \varepsilon^{-1}\left(\tilde{x}^1 - x_*\right), \quad \eta^1 = \varepsilon^{-1}\left(\tilde{q}^1 - \right. \qquad (1.232)$$

are bounded:

$$\xi^1(t) = \varepsilon^{-1}\int_0^T \chi_{1j}(t-s)\left[V\left(q_* + \varepsilon\eta^1\right) - \quad \cdot)\right]ds + \int_0^T \chi_{1j}(t-s)\left[f_*(s) - f^0(s)\right]ds =$$
$$= \varepsilon^{-1}I_1 + I_2, \qquad (1.233)$$

$$\eta^1(t) = -\varepsilon^{-1}\int_0^T \chi_{1j}(s-t)\left[\varphi_x\left(s, x_* + \varepsilon\right. \quad \varphi_x\left(s, x_*\right)\right]ds. \qquad (1.234)$$

For $\varepsilon \to 0$ the first term in (1.233) be presented in the form, analogous to

(1.218):

$$\varepsilon^{-1} I_1(t) = \int_0^T \chi_{1j}(t-s)\left[\sum_{k=1}^{m} U_k \delta(q_*(s) - a_k)\eta^1(s) + \{V_q[q_*(s)]\}\eta^1(s)\right]ds +$$

$$+O\left(\varepsilon(\eta^1)^2\right) = j_1(t) + j_2(t) + O\left(\varepsilon(\eta^1)^2\right), \tag{1.235}$$

where, as also in (1.219),

$$j_1(t) = \sum_{k=1}^{m}\sum_{\nu=1}^{n} U_k \chi_{1j}(t - s_k^\nu)\left|\dot{q}^0(s_k^\nu)\right|^{-1}\eta(s_k^\nu) + O(\varepsilon), \tag{1.236}$$

$$j_2(t) = \int_0^T \chi_{1j}(t-s)\{V_q(q^0(s))\}\eta^1(s)ds + O(\varepsilon).$$

At the same time, owing to (1.221),

$$I_2(t) = \int_0^T \chi_{1j}(t-s)[f_*(s, x_*(s)) - f(s, x^0(s))]ds = \varepsilon\varphi(t) \tag{1.237}$$

for $|f_x(s,x)| \le M$, and, finally,

$$\eta'(t) = -\int_0^T \chi_{1j}(t-s)\rho(s)\xi'(s)ds, \quad \rho(s) \le \max_{x\in X}\varphi_x(s,x). \tag{1.238}$$

Substituting (1.238) in (1.236), (1.235) and accounting for (1.237), we get

$$\xi^1(t) = \int_0^T \kappa(t,s)\xi^1(s)ds + \varepsilon\varphi(t),, \tag{1.239}$$

where the kernel $\kappa(t,s)$ is written as the result of the obvious transformations, and $\kappa(t+T, s*T) = \kappa(t,s)$, $|\kappa(t,s)| \le \kappa < \infty$ and $\varphi(t-T) = \varphi(t)$. As far as there are no resonance in the system, then Eq. (1.239) has the unique periodic solution $\xi^1(t) = \varepsilon\, \mathrm{L}\varphi$. Some methods for calculations of the operator L are given in [115]; let us only note that for any bounded function φ its norm $|\mathrm{L}\varphi| \le C < \infty$, i.e. $|\xi^1(t)| \le C\varepsilon$, and, consequently, owing to (1.238), $|\eta^1(t)| \le c\varepsilon$. Thus,

$$|\tilde{x}^1(t) - x_*(t)| \le c\varepsilon^2, \quad |q'(t) - q_*(t)| \le c\varepsilon^2. \tag{1.240}$$

Employing induction, it is easy to show, that

$$|\tilde{x}^l(t) - x_*(t)| \le c\varepsilon^{l+1}, \quad |\tilde{q}^l(t) - q_*(t)| \le c\varepsilon^{l+1}. \tag{1.241}$$

All the results can be expanded to a multidimensional case.

Further, let $\tilde{u}^l(t)$ be the control found in the l-th approximation, and $\tilde{x}_*^l(t)$ be the solution to (1.194) for $u = \tilde{u}^l(t)$. Then, the control $\tilde{u}^l(t)$ is quasi-optimal with respect to the initial system with estimates $O(\varepsilon^{l+1})$ [compare with (1.215), (1.216)]. This estimates can be proved in the way as it was done in Theorem 1.4.

The obtained results can be formulated in the following way.

Theorem 1.5. *Let the periodic movement of the system* (1.194) *be described by Eq.* (1.205), *where u is a periodic control determined by the minimum condition of the functional* (1.195), *and the conditions of* Theorem 1.4 *are fulfilled. Furthermore, let* $\tilde{u}^l(t)$ *be the control minimizing the functional* (1.195) *on trajectories of the system*

$$\tilde{x}^l(t) = \int_0^T K(t-s)\left[Bu(s) + F(s) + \varepsilon f\left(s, \tilde{x}^{l-1}(s)\right)\right]ds\,, \tag{1.242}$$

where $\tilde{x}^{l-1}(t)$ *is the solution of the preceding optimal problem.*

Then

$$\left|\int_0^T \left[\tilde{u}^l(t) - u_*(t)\right]dt\right| \le c\varepsilon^{l+1}\,, \tag{1.243}$$

$$\left|\tilde{x}_*^l(t) - x_*(t)\right| \le c\varepsilon^{l+1}\,, \tag{1.244}$$

$$\Phi_\varepsilon\left(\tilde{u}^l\right) - \Phi_\varepsilon(u_*) \le c\varepsilon^{l+1}\,,$$

where $x_*^l(t)$ *is the solution of the system* (1.205) *for* $u = \tilde{u}^l(t)$. *If the function V(q) is continuous on the surfaces* (1.200) *and satisfies the Lipschitz condition, then the estimate* (1.243) *can be strengthened, namely,*

$$\left|\tilde{u}^l(t) - u_*(t)\right| \le c\varepsilon^{l+1}\,. \tag{1.245}$$

1.6.3
The Method of Successive Approximations in Problems of the Optimal High-Speed Action

The movement period up to now was considered to be fixed and was determined by the period of the external excitation. Let us show that the schemes of the successive approximations given in Theorems 1.4, 1.5 remain true also for the case when the movement period is not fixed but is determined by optimality conditions.

Theorem 1.6. *Let* $\theta = [T_1, T_2]$ *and*

1) *for any* $T \in \theta$ *there exists the unique isolated T-periodic solution* $\{x_*(t,T), u_*(t,T)\}$ *of the optimal problem* (1.205), (1.195) *treated as the limit of*

the succession $\{x^l(t,T),u^l(t,T)\}$ *(or* $\{\tilde{x}^l(t,T),\tilde{u}^l(t,T)\}$*) for* $l \to \infty$;

2) *for* $l = 0$ *there exists the unique isolated solution* $T = T^0 \in \operatorname{int}\theta$ *of the optimal high–speed–action problem. Then*

$$\left|T_* - T^l\right| \le c\varepsilon^{l+1}, \tag{1.246}$$

where T_* *is an optimal period,* T^l *is a period determined by the l-th approximation;*

3) $x_*^l(\tilde{x}_*^l)$, $T_*^l(\tilde{T}_*^l)$ *are the solution of Eq.* (1.205) *for* $u = u^l$ $(\tilde{u} = \tilde{u}^l)$ *and respective oscillation period. Then the estimates of* Theorems 1.4 *and* 1.5, *and*

$$\left|T_* - T^l\right| \le c\varepsilon^{l+1}, \quad \left|T_* - \tilde{T}^l\right| \le c\varepsilon^{l+1} \tag{1.247}$$

are true.

Proof. The condition

$$s_* = \partial S_* / \partial T = 0 \tag{1.248}$$

serves for the determination of the period T_*. Here

$$S_*(t) = \int_0^T L_*(t,T)dt\,; \tag{1.249}$$

$L_*(t,T)$ is the Lagrange function (1.197) of the problem under study which directly depends on the period T for $u = u_*(t,T)$, $x = x_*(t,T)$. Let

$$S^l(t) = \int_0^T L^l(t,T)dt\,,$$
$$s^l = \partial S^l / \partial T\,, \quad s_T^l = \partial S_T^l / \partial T\,, \tag{1.250}$$

where $L^l(t,T)$ is the Lagrange function (1.197) for $u = u^l(t,T)$, $x = x^l(t,T)$. Then T^l is the solution of the equation

$$s^l(T) = 0\,. \tag{1.251}$$

Let us show, that there exists the limit of the succession $\{T^l\}$ for $l \to \infty$. Let us show at first that all the approximations T^l stay in the ε-zone of the generating solution T^0. Indeed, owing to (1.251)

$$s^l(T^l) = s^0(T^0) = 0\,. \tag{1.252}$$

Then

$$s^l\left(T^l\right)-s^0\left(T^l\right)=s^0\left(T^0\right)-s^0\left(T^l\right). \tag{1.253}$$

In its turn, from Theorems 1.4, 1.5 follows, that

$$\left|s^l\left(T^l\right)-s^0\left(T^l\right)\right|\le c\varepsilon . \tag{1.254}$$

On the other hand,

$$s^0\left(T^l\right)-s^0\left(T^0\right)=s_T^0\left(T^l-T^0\right)+O\left(\left|T^l-T^0\right|^2\right), \tag{1.255}$$

and from the suggestion about the existence of the unique isolated solution T^0 follows that $s_T^0\left(T^0\right)\ne 0$. Comparing (1.253)-(1.255), we get

$$\left|T^l-T^l\right|\le c\varepsilon . \tag{1.256}$$

for $s_T^0\left(T^0\right)\ne 0$.

We now compare approximations T^l and T^{l-1}. In any approximation

$$s^{l-1}\left(T^{l-1}\right)=s^l\left(T^l\right)=0 .$$

Taking into account, that for all $T\in\theta$

$$\left|s^l(T)-s^{l-1}(T)\right|\le c\varepsilon^{l+1}$$

and writing

$$s^{l-1}\left(T^l\right)-s^{l-1}\left(T^{l-1}\right)=s_T^{l-1}\left(T^{l-1}\right)\left(T^l-T^{l-1}\right)+O\left(\left|T^l-T^{l-1}\right|^2\right),$$

we get

$$\left|T^l-T^{l-1}\right|\le c_1\varepsilon^{l+1}\left[s_T^{l-1}\left(T^{l-1}\right)\right]^{-1}.$$

At the same time, accounting for the estimates (1.254), (1.256), we have

$$s_T^{l-1}\left(T^{l-1}\right)=s_T^0\left(T^0\right)+O(\varepsilon).$$

If T^0 is the unique isolated root of the equation $s^0\left(T^0\right)=0$, i.e., $s_T^0\left(T^0\right)\ne 0$, then $s_T^{l-1}\left(T^{l-1}\right)\ne 0$, and

$$\left|T^l-T^{l-1}\right|\le c\varepsilon^{l+1}. \tag{1.257}$$

Thus, for sufficiently small ε the series $T^0+\sum_{l=1}^{\infty}\left(T^l-T^{l-1}\right)$ converges absolutely and uniformly to the limit T_∞. By definition, T_∞ is the solution of the equation

$s_\infty(T) = \lim_{l\to\infty} s^l(T) = s_*(T)$. Hence, $T_\infty = T_*$. Thus,

$$\left|T_* - T^l\right| = \left|\sum_{r=l}^{\infty}\left(T^{r+1} - T^r\right)\right| \le c\varepsilon^{l+1}. \tag{1.258}$$

The estimate (1.247) is proved in the same way.

2 Periodic Control for Vibroimpact Systems

Impact problems arise in various application fields. The problems of systematic impacts are of special interest. Systems in which systematic impacts between separate elements, or between elements and stiff limiters, are realized are called *vibroimpact systems*.

An impact is a basis of technological processes in impact–action mechanisms, and it is the constructor's aim to realize the system with the maximal impact intensity.

At the same time, in mechanical systems there are often undesirable vibroimpact processes are caused by impacts against limiters, by impacts during the motion in clearances, etc. In the analysis of such phenomena, the questions of the existence of vibroimpact regimes come to the fore.

The processes which occur at impact of bodies are sufficiently complicated, and there are many models of impact [16, 71, 72, 108]. Still, the utilization of a stereomechanical model, according to which the impact is considered to be momentary, is often possible Such an idealization holds for a majority of mechanical and controlled systems, characteristic times of which considerably overcome a duration of the contact of bodies at impact. Thus, in the approximation of the momentary impact, the elastic properties of elements at impact are neglected, and differential motion equations are supplemented by finite relations characterizing the velocity discontinuity at the impact moment.

A diversity of models of vibroimpact systems and specific features of their description demanded an elaboration of respective mathematical tools. For a long time a method of point mapping [106] remained the main investigation tool. This exact method is based on the adding of impactless solutions divided by the impact moment. Results obtained in this direction are reflected in monographs [71, 72].

Obviously, the adding method is suitable in the first place for an analytical study of linear systems of low dimension. If the system is non-linear, then an attempt to construct an analytical solution for the period between impacts usually fails; if the system is linear but has a high order, then the computational difficulties appear linked with the necessity of determination of a large number of integration constants which are a part of adding conditions.

A method of periodic Green's functions appeared to be an effective analysis method for periodic regimes of vibroimpact systems [16]. It was shown in Section 1, that the motion description by means of integral equations allows us to remove many of the difficulties generated by the high order of the system and to single out the motion for one or several coordinates.

Such a separation in analysis of vibroimpact systems is naturally linked with the presence of striking elements. The technique of application of the method of periodic Green's functions for analysis of vibroimpact systems was described in [23]. This method got the further development in [16].

Approximation methods of analysis of vibroimpact systems are also advantageous. A method of harmonic linearization of impact non-linearities [16] provides in many cases a sufficient qualitative and quantitative accuracy of its results, though the problem of the accuracy for a general case remains open.

An analysis of vibroimpact systems is constructed on the basis of asymptotic methods [20, 24, 47, 50, 51, 57, 58]. Suggested non-smooth substitutions of variables allow the reduction of the motion equations to a standard form and the use of an averaging method. This way is useful for the study of systems close to conservative ones, and, in particular, for an analysis of steady oscillations in vibroimpact systems.

A detailed review of analysis methods of dynamics of vibroimpact systems is given in [16, 24, 71, 72]. The main ideas of the method of integral equations and its application for analysis and synthesis of controlled vibroimpact systems are given in Sections 2.1, 2.2.

Problems of an optimal control of vibroimpact systems, or, in a more general case, of systems with velocity discontinuities were analyzed in [12-16, 19, 73, 75, 125, 162, 189]. A solution was obtained, as a rule, with the use of the maximum principle: A motion was described by means of differential equations, and the conditions of periodicity and of impact were treated as additional constraints linking coordinates and velocities of system's points.

An approach based on the description of the periodic motion of a system with the use of integral equations seems to be more effective.

The main characteristics of a vibroimpact system are impulse and frequency of impacts which depend, in their turn, on the motion of a striking element. The method of integral equations allows us to single out one equation describing the motion of the striking element; the system's structure determines the form of the kernel - the periodic Green's function. Main restrictions of the problem are dimensions of the structure and the energy source power. These demands can be reduced in the mainly used structures to a modulus control restriction (restriction of dimensions in pneumatic or hydraulic systems), or to a mean-square restriction (the structure weight reduction linked with reduction of the compressor's capacity, etc.). These construction restrictions are analyzed in more details in [15, 16].

Thus, the problem can be formalized in the form of a single integral equation that presupposes effectiveness of optimization methods given in Section 1.3.

The exact solution for an optimal control of a system with one striking element, linear between impacts is given in Section 2.3 [73]. An optimal one-impact regime is found. It is obvious, that such a regime is mostly advantageous with respect to energy [16]; an optimality of the one-impact regime for a system with one degree of freedom is proved in [62].

Problems of a choice of an optimal period and an optimal clearance (press fit) are solved together with a search for an optimal control.

In Section 2.4 approximate methods for a solution of an optimal control problem in systems with additional non-linear and non-stationary constraints are given. The main attention is paid to a control of resonant systems as far as for the system in a resonant regime a weak control excitation can lead to a considerable effect. A problem of realization of an optimal control arises in practical application of obtained results; some possibilities of realization based on accepted schemes of driving mechanisms of force impulse systems are discussed in [15].

2.1 Motion Equations of Vibroimpact Systems. Integral Equations of Periodic Motions

Let us construct the motion equations for a vibroimpact system under the consideration of the momentary impact.

At the direct central impact of two bodies with masses M_1, M_2 moving with velocities $\dot{x}_1$, $\dot{x}_2$ (Fig. 2.1) we have, owing to the conservation law of momentum,

$$M_1\dot{x}_{1-} + M_2\dot{x}_{2-} = M_1\dot{x}_{1+} + M_2\dot{x}_{2+} . \tag{2.1}$$

Here a minus sign designates the velocities for the impact, while a plus sign designates velocities after it. A linkage between the velocities before and after the impact can be expressed by the relation

$$|\dot{x}_{2+} - \dot{x}_{1+}| = R|\dot{x}_{2-} - \dot{x}_{1-}| , \tag{2.2}$$

where R is a coefficient of velocity recovery at impact, $0 \le R \le 1$. The greater R, the greater energy loss at impact; $R = 1$ corresponds to an absolutely elastic impact, $R = 0$ to a plastic one.

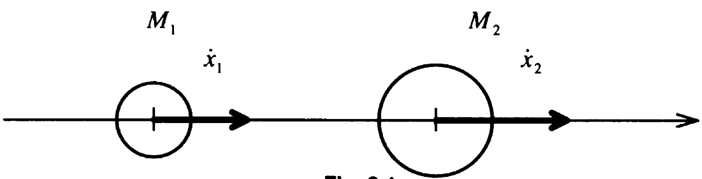
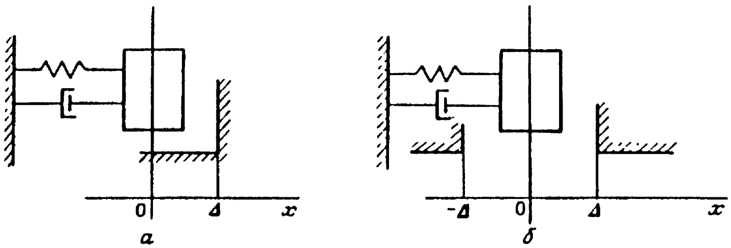


Fig. 2.1

A case of an impact against a fixed limiter $(\dot{x}_2 = 0)$ (Fig. 2.2.a) is of special interest. In this case at the impact moment

$$x = \Delta , \qquad \dot{x}_+ = -R\dot{x}_- . \tag{2.3}$$



a b

Fig. 2.2

A positive direction for velocity is considered below in the same way as in [16]:

$$\dot{x}_- \geq 0 . \qquad (2.4)$$

$\Delta > 0$ corresponds to a system with a clearance, while $\Delta < 0$ corresponds to a system with a press fit. Possible vibroimpact regimes satisfy the condition of the absence of additional intersections

$$x(t) \leq \Delta . \qquad (2.5)$$

This condition corresponds to an assumption that a motion trajectory does not intersect a level of the limiter between two calculated impacts.

For a system with two symmetric limiters (Fig. 2.2.b), condition (2.5) is reduced to the form

$$|x(t)| \leq \Delta . \qquad (2.6)$$

Here, the velocity value at the impact against the right limiter is considered to be positive according to (2.4).

The value $J = M|\dot{x}_- - \dot{x}_+|$ is called an impact *impulse* (M is a mass of the body). Owing to (2.3), the impulse is determined by the relation

$$J = M(1 + R)|\dot{x}_-| . \qquad (2.7)$$

An analogous characteristic can be constructed also for a case of an impact of two bodies. If x is a relative coordinate of two bodies (Fig. 2.1) $(x = x_1 - x_2)$, then the value

$$J = M|\dot{x}_- - \dot{x}_+| \qquad (2.8)$$

is called an impulse, where $M = M_1 M_2 / (M_1 + M_2)$ is a reduced mass of striking bodies, $\dot{x}_-$ and $\dot{x}_+$ are relative velocities for and after the impact, respectively.

Let us write the motion equations of the system with one striking element and the fixed stiff limiter. An equation of the motion between two impacts for the striking element with the mass M can be written in the form

$$M\ddot{x} = P(t,x). \tag{2.9}$$

Here $P(t,x)$ is a force applied to the striking element, M is its mass.

Eq. (2.9) is supplemented by the impact condition

$$t = t_*, \quad x = \Delta, \quad \dot{x}_+ - \dot{x}_- = -(1+R)\dot{x}_- = -J_*/M. \tag{2.10}$$

Here t_* is the impact moment, Δ is a clearance (press fit), J_* is the impact impulse. Let us substitute a discontinuity condition at the impact moment into the motion equation.

As known [70], a discontinuity of derivative containing the delta function

$$\dot{f}_+(t_*) - \dot{f}_-(t_*) = s\delta(t - t_*) \tag{2.11}$$

corresponds to the discontinuity of the first kind of the function $f(t)$ in the point t_*

$$f_+(t_*) - f_-(t_*) = s. \tag{2.12}$$

Integrating (2.11) over the interval from $t_* - 0$ to $t_* + 0$, and accounting for the properties of the delta function [43, 70], we get (2.12).

Thus, the discontinuity of acceleration containing the delta function corresponds to the finite discontinuity of velocity. Accounting for (2.10)–(2.12) and substituting the impact conditions into the motion equation, we get

$$M\ddot{x} = P(t,x) - J_*\delta(t - t_*). \tag{2.13}$$

Thus, the impact can be treated as a force interaction, and the value

$$\Phi = -J_*\delta(t - t_*) \tag{2.14}$$

can be considered as one of acting forces. The force characteristics of an impact interaction are constructed in [23] as non-linear functions of coordinates and velocities, equivalent to (2.10), (2.14).

A replacement of relations (2.9), (2.10) with Eq. (2.13) is especially effective in analysis of a periodic motion of vibroimpact systems.

Consider a system, linear during periods between impacts (Fig. 2.3), with one striking element A of mass M. Periodic forces $P_t(t) = P_t(t+T)$ can generate in the system with the limiter both periodic regimes with the period T and sub-harmonic regimes with the period equal to a multiple of the excitation period. Besides regimes with one impact during the period, multi-impact regimes which are characterized by several impacts against the limiter during a period are also possible in vibroimpact systems. One-impact T-periodic regimes have the most

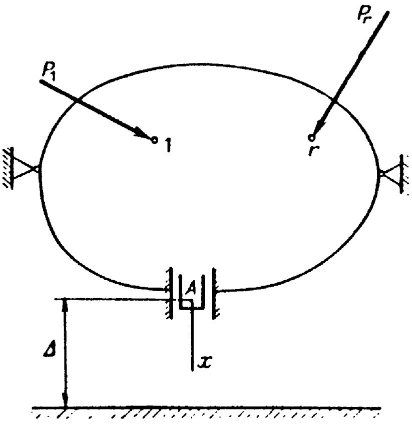


Fig. 2.3

intensive (resonant) character.

Below we will pay the main attention to regimes of such a kind.

Let us write an equation for the one-impact T-periodic regime. The impact conditions (2.10) are reduced to the form

$$t = kT + \varphi\,, \qquad x = \Delta\,, \qquad (2.15)$$
$$\dot{x}_+ - \dot{x}_- = -J/M\,.$$

Here x is a coordinate of the element A, $J_. \geq 0$ is an impact impulse, φ is an impact phase. J and φ are constant values for a periodic regime.

Let $l_A^1(p)$,..., $l_A^n(p)$ be operators of dynamic compliance linking displacements of the point A with forces applied in points 1,..., n, and $l(p)$ be an operator of dynamic compliance in the point A. Treating the impact as the force interaction (2.14) and substituting the discontinuity condition (2.15) into the motion equation, we get the equation of periodic regime

$$x = \sum_{m=1}^{n} l_A^m(p)P_m(t) - l(p)\left[J\delta^T(t-\varphi)\right]. \qquad (2.16)$$

Here $\delta^T(t)$ is a T-periodic delta function which is expressed by the series (1.25), (1.25):

$$\delta^T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{ki\omega t} = \sum_{k=-\infty}^{\infty} \delta(t-kT)\,.$$

The unknown impulse J and the impact phase φ are determined by the impact conditions

$$x(\varphi) = \Delta\,, \qquad J = M(1-R)\dot{x}_-(\varphi)\,, \qquad (2.17)$$

where M is the reduced mass of striking elements.

The periodic solution of the system (2.16) has the form (see Section 1.2)

$$x(t) = \int_0^T \left[\sum_{m=1}^{n} \chi_1^m(t-s)P_m(s) - \chi_1(t-s)J\delta^T(s-\varphi)\right] ds\,, \qquad (2.18)$$

where χ_1^m and χ_1 are periodic Green's functions of the first kind corresponding to the operators $l_A^m(p)$, $l_A(p)$:

$$\chi_1^m(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty} l_A^m(ki\omega)e^{ki\omega t},$$

$$\chi_1(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty} l(ki\omega)e^{ki\omega t}.$$

Using properties of delta function, let us write the solution (2.18) in the form:

$$x(t)=\theta_1(t)-J\chi_1(t-\varphi), \tag{2.19}$$

where $\theta_1(t)$ is a periodic solution corresponding to the motion without the limiter, which can be expressed by the first term in (2.18); the second term in it is the system's response to the periodic sequence of impact impulses.

Applying the impact conditions (2.17) to (2.19), we get the system of transcendent equations which determine J and φ

$$\theta_1(\varphi)=\Delta+J\chi_1(0), \tag{2.20}$$

$$J=\dot{\theta}_1(\varphi)\left[\dot{x}_{1-}(0)+(1+R)^{-1}M\right]^{-1}.$$

In a particular case of the excitation by one harmonic force $P_1(t)=P_0\cos\omega t$, $P_2(t)=....P_n(t)=0$, the solution of a linear problem has the form

$$\theta_1(t)=a\cos\omega(t+\psi), \tag{2.21}$$

where

$$a=|l(i\omega)|P_0, \qquad \tan\omega\psi=\operatorname{Im}l(i\omega)/\operatorname{Re}l(i\omega).$$

Then

$$\theta_1(\varphi)=a\cos\omega(\varphi+\psi)=a\cos\omega\tilde{\varphi}, \qquad \tilde{\varphi}=\omega(\varphi+\psi), \tag{2.22}$$

$$\dot{\theta}_1(\varphi)=-\omega a\sin\omega(\varphi+\psi)=-\omega a\sin\omega\tilde{\varphi}.$$

Solving the system (2.20)–(2.22) with respect to J, $\tilde{\varphi}$, we get

$$J=\frac{-\Delta\chi_1(0)\pm\sqrt{\Delta^2\chi_1^2(0)+\dfrac{a^2-\Delta^2}{\omega^2}\left[\dot{\chi}_{1-}(0)+\dfrac{1}{M(1+R)}\right]^2}}{\chi_1^2(0)+\dfrac{1}{\omega^2}\left[\dot{\chi}_{1-}(0)+\dfrac{1}{M(1-R)}\right]^2}, \tag{2.23}$$

$$\cos\tilde{\varphi}=\frac{\Delta+J\chi_1(0)}{a}, \qquad J\geq 0.$$

The found solution (2.23) should be proved with respect to stability [16] and the absence of additional intersections: $x(t) \le \Delta$.

If non-linear reactions – stationary and non-stationary – are also included into excitation forces, then the motion equations

$$x = l_0(p)y, \quad y = -\mu G(t,x) \tag{2.24}$$

and the impact conditions do not have, as a rule, an analytical solution. The respective integral equation of the periodic regime

$$x(t) = -\mu \int_0^T \chi_{10}(t-s)G(s,x(s))ds - J\chi_{10}(t-\varphi), \tag{2.25}$$

where $\chi_{10}(t)$ is a periodic Green's function of the linear part of the system, can be solved either numerically or approximately.

Some schemes of approximate solution of equations of the type (2.25) are given in Section 2.4.

In the same way we can write equations of periodic regimes in systems with double-sided striking pair for impacts against limiter in turn. Suppose, for the sake of simplicity, that during the periods between impacts the system is linear and stationary, and let us study various cases of positions of limiters.

1. Acting forces change their size after a half-period, $P_m(t) = -P_m(t+T/2) = P_m(t+T)$, $m = 1,\dots, n$, the limiters are symmetric $\Delta_1 = \Delta_2 = \Delta$ (Fig. 2.2.b), and recovery coefficients $R_1 = R_2 = R$. Obviously, in this case

$$\begin{aligned} x(t) &= -x(t+T/2) = x(t+T), \\ \dot{x}(t) &= -\dot{x}(t+T/2) = \dot{x}(t+T), \end{aligned} \tag{2.26}$$

and, owing to symmetry, the impact conditions can be written in the form

$$\begin{aligned} &t = \varphi, \quad x = \Delta, \quad \dot{x}_+^{(1)} = -R\dot{x}_-^{(1)}, \\ &t = \varphi + T/2, \quad x = -\Delta, \quad \dot{x}_+^{(2)} = -R\dot{x}_-^{(2)}, \quad \dot{x}_-^{(2)} = -\dot{x}_-^{(1)}. \end{aligned} \tag{2.27}$$

Hence, the impact impulses against the right and left limiters

$$\begin{aligned} J_1 &= M\left(\dot{x}_-^{(1)} - \dot{x}_+^{(1)}\right) = (1+R)M\dot{x}_-^{(1)}, \\ J_2 &= M\left|\dot{x}_-^{(2)} - \dot{x}_+^{(2)}\right| = (1+R)M\left|\dot{x}_-^{(2)}\right| = (1+R)M\dot{x}_-^{(1)} \end{aligned} \tag{2.28}$$

are equal:

$$J_1 = J_2 = J. \tag{2.29}$$

Treating the impact as the force interaction (2.14) and substituting the condition of velocity discontinuity into the motion equations, we get, accounting for the change of the impact interaction sign,

$$x(t) = \sum_{m=1}^{n} l_A^m(p)P_m(t) - l(p)\left[J\delta^T(t-\varphi) - J\delta^T(t-T/2-\varphi)\right]. \tag{2.30}$$

According to (1.48), define

$$\delta_2^T(t) = \delta^T(t) - \delta^T(t - T/2), \tag{2.31}$$

where $\delta_2^T(t)$ is the periodic delta function of the second kind. Accounting for (2.26), we re-write the periodic reaction of the system (1.30) in the form

$$x(t) = \theta_2(t) - J\chi_2(t - \varphi), \quad 0 < t < T/2, \tag{2.32}$$

where $\theta_2(t)$ is a periodic reaction of the system without the limiters [see (1.47)],

$$\theta_2(t) = \sum_{m=1}^{n} \int_0^{T/2} \chi_2^m(t-s) P_m(s) ds, \tag{2.33}$$

and $\chi_2^m(t)$ and $\chi_2(t)$ are the periodic Green's functions of the second kind corresponding to operators $l_A^m(p)$, $l(p)$:

$$\chi_2^m(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} l_A^m(ki\omega) e^{ki\omega t},$$

$$\chi_2(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} l(ki\omega) e^{ki\omega t}.$$

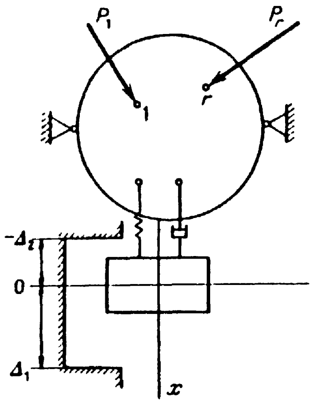


Fig. 2.4

Substituting (2.32) into (2.27), (2.28), we get the system of transcendental equations determining an impulse and phase of the impact:

$$\theta_2 = \Delta + J\chi_2(0),$$
$$J = \dot\theta_2(\varphi)\left[\dot x_{2-}(0) + \left[M(1+R)^{-1}\right]^{-1}\right], \tag{2.34}$$

which coincides – up to the index – with (2.20).

2. The limiters of the system have arbitrary positions, $\Delta_2 \neq \Delta_1$, $R_2 \neq R_1$ (Fig. 1.4). Then the impact conditions get the following form

$$t = \varphi_1, \quad x = \Delta_1, \quad \dot x_+^{(1)} = -R_1 \dot x_-^{(1)}, \tag{2.35}$$
$$t = \varphi_2 + T/2, \quad x = -\Delta_2, \quad \dot x_+^{(2)} = -R_2 \dot x_-^{(2)}.$$

The impact impulses are determined by formulas

$$J_1 = M\left(\dot x_-^{(1)} - \dot x_+^{(1)}\right) = (1 + R_1) M \dot x_-^{(1)}, \tag{2.36}$$

$$J_2 = M\left(\dot{x}_-^{(2)} - \dot{x}_+^{(2)}\right) = -(1 + R_2)M\dot{x}_-^{(2)} .$$

From the motion equations accounting for the force characteristics of the impact interaction

$$x = \sum_{m=1}^{n} l_A^m(p)P_m(t) + l(p)\left[-J_1\delta^T(t-\varphi_1) + J_2\delta^T(t-\varphi_2)\right], \tag{2.37}$$

we get the periodic motion equation

$$x(t) = -J_1\chi_1(t-\varphi_1) + J_2\chi_1(t-\varphi_2) + \theta_1(t), \tag{2.38}$$

where $\theta_1(t)$ is a periodic reaction of the system without the limiters [see (2.18)]. Substituting (2.38) into (2.35), (2.36), we get

$$\dot{x}_-^{(1)} = -J_1\dot{\chi}_{1-}(0) + J_2\dot{\chi}_1(\varphi_1 - \varphi_2) + \dot{\theta}_1(\varphi_1) = J_1\left[M(1+R_1)\right]^{-1},$$

$$\dot{x}_-^{(2)} = -J_1\dot{\chi}_1(\varphi_2 - \varphi_1) + J_2\dot{\chi}_{1-}(0) + \dot{\theta}_1(\varphi_2) = -J_2\left[M(1+R_2)\right]^{-1},$$

or,

$$\mu_1 J_1 - \dot{\chi}_1(\varphi_1 - \varphi_2)J_2 = \dot{\theta}_1(\varphi_1), \tag{2.39}$$

$$\mu_2 J_2 + \dot{\chi}_1(\varphi_2 - \varphi_1)J_1 = \dot{\theta}_1(\varphi_2),$$

where

$$\mu_j = J_1\left[(1+R_j)M\right]^{-1} + \dot{\chi}_{1-}(0) \tag{2.40}$$

and

$$-J_1\chi_1(0) + J_2\chi_1(\varphi_1 - \varphi_2) + \theta_1(\varphi_1) = \Delta_1, \tag{2.41}$$

$$-J_1\chi_1(\varphi_2 - \varphi_1) + J_2\chi_1(0) + \theta_1(\varphi_2) = -\Delta_2 .$$

Relations (2.39)–(2.41) serve for determination of impulses J_1, J_2 and phases φ_1, φ_2 of the impacts. In particular, if

$$\Delta_1 = \Delta_2 = \Delta, \quad R_1 = R_2 = R, \quad \theta_1(t + T/2) = -\theta_1(t),$$

then Eqs. (2.39)–(2.41) have the solution $J_1 = J_2 = J$, $\varphi_2 - \varphi_1 = T/2$, and can be reduced by means of the obvious transformations to the system (2.34). Let us note that if the limiters are not symmetric, but $P_m(t) \neq -P_m(t + T/2)$, then the motion is also asymmetric, $J_1 \neq J_2$, $\varphi_1 \neq \varphi_2$, and for the determination of the impulses and phases of impacts the system (2.39)–(2.41) should be used with substitution of $\mu_1 = \mu_2 = \mu$, $\Delta_1 = \Delta_2 = \Delta$.

If there are no dissipation in the system, $\mu_1 = \mu_2 = 0$, and

$$\chi_1^0 = \frac{1}{T} + \frac{2}{T}\sum_{k=1}^{\infty} l_0(k\omega)\cos k\omega t\,,$$

$$l_0(k\omega) = \operatorname{Re} l(ki\omega)\,, \qquad \operatorname{Im} l(ki\omega) = 0\,, \tag{2.42}$$

then, obviously,

$$\chi_1^0(t) = \chi_1^0(-t)\,, \qquad \dot{\chi}_1^0(T/2) = 0\,. \tag{2.43}$$

Let $\theta_1(t) = -\theta_1(t + T/2)$, $\dot{\theta}_1(t) = -\dot{\theta}_1(t + T/2)$. Then the impact phases φ_1 and $\varphi_2 = \varphi_1 + T/2$ are determined by the equation

$$\dot{\theta}_1(\varphi_1) = 0\,, \tag{2.44}$$

and for calculation of the impulses J_1, J_2 the system of equations

$$-J_1\chi_1(0) + J_2\chi_1(T/2) = \Delta_1 - \theta_1(\varphi_1)\,,$$

$$-J_1\chi_1(T/2) + J_2\chi_1(0) = -\Delta_2 + \theta_1(\varphi_2) \tag{2.45}$$

can be used, but $J_1 \neq J_2$ for $\Delta_1 \neq \Delta_2$.

2.2 Resonant and Quasi-Resonant Oscillations of Vibroimpact Systems

Specific features of a periodic motion of mechanical systems are especially apparent in analysis of near-resonant regimes. Intensive oscillations in vicinity of the resonance are supported by a small exciting force, and the motion can be treated as close to a free one in a case when dissipative forces in the system are small. In this connection the main attention is paid to a study of free oscillations of a conservative system. Using such dynamic model, it is possible to reveal the main properties of the oscillatory system in a one-impact periodic regime.

2.2.1 Oscillations of Conservative Systems with One Degree of Freedom

a) *Oscillations of a system with a one-sided limiter.* Let us examine free oscillations of a linear conservative system with one degree of freedom and a one-sided limiter (Fig. 2.2.a for $b = 0$) for an absolutely elastic impact $(R = 1)$. Consider the impact moment to be a begin of the time axis, and let us account for conditions of impact and periodicity

$$x(0) = x(T) = \Delta\,, \qquad \dot{x}(0_+) = -\dot{x}(T_-) = -v_0 \tag{2.46}$$

in a solution of free oscillation equation

$$\ddot{x} + \Omega^2 x = 0, \quad \Omega^2 = c/M, \quad 0 < t < T \tag{2.47}$$

(below $M = 1$ is considered). Here $T = 2\pi/\omega_0$ is a period of free oscillations of an impact oscillator, ω_0 is an oscillation frequency.

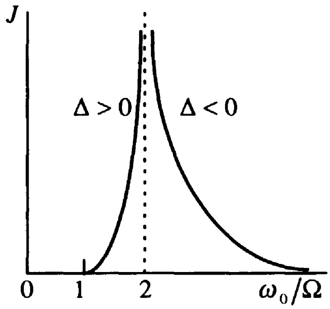


Fig. 2.5

In a period between two impacts a motion law corresponds to the response of the system to a periodic sequence of impulses:

$$x = -J\chi_1(t), \tag{2.48}$$

where $J = 2v_0$ is an impact impulse, $\chi_1(t)$ is a periodic Green's function of the first kind of the system (2.47)

$$\chi_1(t) = \frac{1}{2\Omega} \frac{\cos\Omega(t - T/2)}{\sin\Omega T/2}, \quad 0 < t < T. \tag{2.49}$$

The impact impulse is determined by an initial energy reserve of the system; the relation $\omega_0(J)$ can be found from (2.48) with account for the first impact condition (2.46)

$$-J\chi_1(0) = \Delta. \tag{2.50}$$

Substituting (2.49) into (2.50), we get

$$J = -2\Omega\Delta \tan \pi\Omega/\omega_0. \tag{2.51}$$

The graph of (2.51) is shown in Figure 2.5. A demand $J \geq 0$ singles out frequency domains

$$\begin{aligned} &1 < \omega_0/\Omega < 2, \quad \Delta > 0; \\ &\omega_0/\Omega > 2, \quad \Delta < 0. \end{aligned} \tag{2.52}$$

Figure 2.6.a–c presents phase diagrams of the system under study which illustrate (2.51). A mapping point moves along the truncated circle with a constant angular velocity Ω. A radius of the circle $\rho = \left(\Delta^2 + v_0^2/\Omega^2\right)^{1/2}$ is fixed by the initial conditions (2.46). As far as the impact is momentary, a time of transition from the point A to the point B is neglected, and the cycle duration – the oscillation period $T_0 = 2\pi/\omega_0$ – is equal to the time of the run along the arc.

In a conservative system without the limiter, with the same initial conditions

(2.46), a mapping point covers the entire circle of the same radius ρ with the same angular velocity Ω during the period $T = 2\pi/\omega_0$. Hence, $T > T_0$,

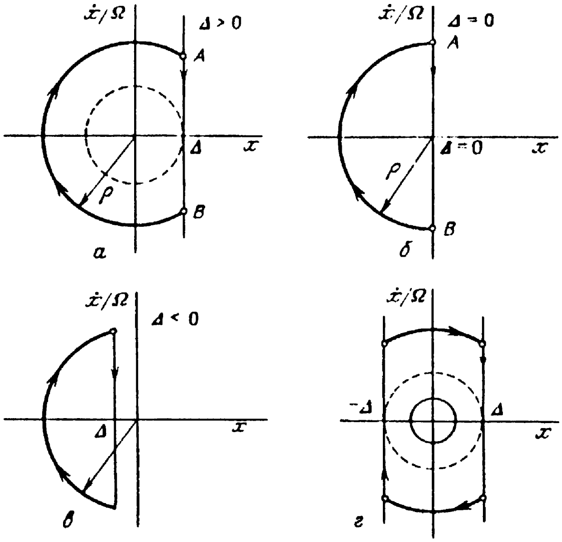


Fig. 2.6

$$\omega_0 > \Omega ,$$

i.e., an introduction of the limiter makes the system more "stiff". Moreover, from Fig. 2.6.b follows

$$\begin{aligned} &1 < \omega_0/\Omega < 2 , \quad \Delta > 0 ; \\ &\omega_0/\Omega = 2 , \quad \Delta = 0 ; \\ &\omega_0/\Omega > 2 , \quad \Delta < 0 , \end{aligned} \tag{2.53}$$

that is in agreement with the condition $J \geq 0$.

In a system with a clearance, an increase in the impact velocity v_0 causes the growth of the oscillation frequency, and vibroimpact non-linearity is "stiff", in a system with a press fit ($\Delta < 0$) the oscillation frequency decreases with the impulse increase, i.e., the non-linearity is "soft". For $\Delta = 0$ the system is isochronous, $\omega_0/\Omega = 2$.

b) *Oscillations of a system with symmetric limiters.* It is not complicate to get analogous relations for a case of a free oscillator with two symmetric limiters (Fig. 2.6.b).

We correlate the begin of the time axis to the moment after the impact against the right limiter, and impose on the equation solution the following conditions:

$$t = 0, \quad x = \Delta, \quad t = T/2, \quad x = -\Delta,$$
$$\dot{x}(0_+) = -\dot{x}(T/2) = -v_0. \tag{2.54}$$

Thus, the system motion corresponds to its response to the stream of impulses, which change their sign after a half-period [see (1.48)], i.e., in the intervals between the impacts

$$x = -J\chi_2(t), \quad 0 < t < T/2, \quad T = 2\pi/\omega_0. \tag{2.55}$$

Here $\chi_2(t)$ is a periodic Green's function of the second kind (2.45)

$$\chi_2(t) = \frac{1}{2\Omega}\frac{\sin\Omega(t - T/\Delta)}{\cos\Omega T/4}, \quad 0 < t < T/2; \tag{2.56}$$

the dependence of the frequency ω_0 on the impulse is given by the impact condition

$$-J\chi_2(0) = \Delta. \tag{2.57}$$

Substituting (2.46) into (2.47), we get

$$J\tan\frac{\pi\Omega}{2\omega_0} = 2\Omega\Delta. \tag{2.58}$$

A demand $J \geq 0$ singles out a frequency domain of free oscillations

$$\omega_0/\Omega \geq 1, \tag{2.59}$$

that also evidently follows from the phase diagram analysis of the system (Fig. 2.6.d). The skeleton graph – the dependence $\omega_0(J)$ – is shown in Figure 2.7. The boundary point $\omega_0 = \Omega$ corresponds to the contact point for $J = 0$.

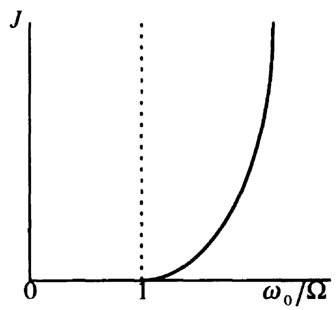


Fig. 2.7

The detailed motion analysis of the system with one-sided and symmetric double-sided limiters is given in [16, 24, 72].

c) *Oscillations of a system with asymmetric limiters.* Let us construct skeleton curves characterizing the dependence of the frequency of free oscillations on an impulse in a system with asymmetric limiters. Correlating the begin of the time axis to the moment of the impact against the right limiter, we will write the impact conditions in the form

$$t=0\,,\qquad x=\Delta_1\,,\qquad \dot{x}(0_+)=-v_0$$
$$t=T/2\,,\qquad x=-\Delta_2\,. \tag{2.60}$$

A velocity level at moment $t=T/2$ is determined by Eq. (2.45) (for $\theta_1=0$). We have

$$-J_1\chi_1(0)+J_2\chi_1(T/2)=\Delta_1\,,$$
$$-J_1\chi_1(T/2)+J_2\chi_1(0)=-\Delta_2\,, \tag{2.61}$$

where $J_1=2v_0$, an impulse $J_2= -2\dot{x}_-(T/2)$ is determined from the solution of the system (2.61) and can be expressed by J_1. From (2.61)

$$J_1=-\frac{1}{2}\left[(\Delta_1+\Delta_2)\chi_2^{-1}(0)+(\Delta_1-\Delta_2)\chi_3^{-1}(0)\right]=j_1-j_2\,, \tag{2.62}$$

$$J_2=-\frac{1}{2}\left[(\Delta_1+\Delta_2)\chi_2^{-1}(0)-(\Delta_1-\Delta_2)\chi_3^{-1}(0)\right]=j_1+j_2\,, \tag{2.63}$$

where still

$$\chi_2(t)=\chi_1(t)-\chi_1(t+T/2),\qquad \chi_2(0)=-(2\Omega)^{-1}\tan\pi\Omega/2\omega_0\,,$$

$$\chi_3(t)=\chi_1(t)+\chi_1(t+T/2),\qquad \chi_3(0)=(2\Omega)^{-1}\cot\pi\Omega/2\omega_0\,.$$

The function $\chi_3(t)$ can be treated as a periodic Green's function of the first kind with the period $T/2$. Eqs. (2.62), (2.63) have the following physical sense. The first term j_1 corresponds to the impact impulse in a conservative system with the limiters situated symmetrically, at $x=\pm(\Delta_1+\Delta_2)/2$, the second term j_2 characterizes the impulse change caused by asymmetry of the system.

From (2.62), (2.63) we have

$$J_1=\Omega(\Delta_1+\Delta_2)\cot\frac{\pi\Omega}{2\omega_0}-\Omega(\Delta_1-\Delta_2)\tan\frac{\pi\Omega}{2\omega_0}\,. \tag{2.64}$$

$$J_2=\Omega(\Delta_1+\Delta_2)\cot\frac{\pi\Omega}{2\omega_0}+\Omega(\Delta_1-\Delta_2)\tan\frac{\pi\Omega}{2\omega_0}\,. \tag{2.65}$$

A demand for existence of a double-impact asymmetric regime $J_1\ge 0$, $J_2\ge 0$ singles out the following domains.

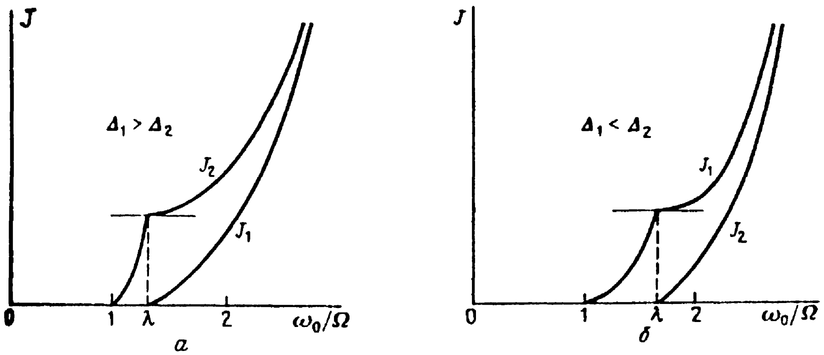
a) $\Delta_1>\Delta_2$. From (2.64), (2.65) follows that $J_2>0$ for $\omega_0/\Omega>1$, the impact against the right limiter exists only for

$$\tan^2 \frac{\pi\Omega}{2\omega_0} \leq \frac{\Delta_1 + \Delta_2}{\Delta_1 - \Delta_2} = \mu^2 , \tag{2.66}$$

and the boundary of the double-impact regime is determined by relation

$$\lambda = \frac{\omega_0}{\Omega} = \frac{\pi}{2 \arctan \mu} , \tag{2.67}$$

i.e., $1 \leq \lambda \leq 2$. The value $\lambda = 1$ corresponds to the symmetric limiter, the value $\lambda = 2$ corresponds to the case $\Delta_2 = 0$, $J_1 = 0$ [quasi-isochronous system, comp. (2.53)]. Eqs. (2.64)–(2.67) are presented in Fig. 2.8.a. For $1 < \omega_0/\Omega < \lambda$ there can exist the regime with impacts only against the left limiter.



a b

Fig. 2.8

b) $\Delta_1 < \Delta_2$. From (2.64), (2.65) follows that $J_1 > 0$ for $\omega_0/\Omega \geq 1$, and the condition $J_2 \geq 0$ is fulfilled , if

$$\tan^2 \frac{\pi\Omega}{2\omega_0} \leq \frac{\Delta_1 + \Delta_2}{|\Delta_1 - \Delta_2|} = \mu^2 . \tag{2.68}$$

The boundary for existence of the double-impact regime is determined by the relation analogous to (2.67):

$$\lambda = \frac{\omega_0}{\Omega} = \frac{\pi}{2 \arctan \mu} . \tag{2.69}$$

The graphs of Eqs. (2.64), (2.65) for $\Delta_1 < \Delta_2$ are presented in Fig. 2.8.b. For $1 < \omega_0/\Omega < \lambda$ there can exist a regime with impacts only against the left limiter.

Thus, depending on the initial energy supply and position of the limiters, regimes with impacts against either one limiter (situated closer) or both limiters can exist in asymmetric system.

2.2.2 Resonant Oscillations of Systems with Several Degrees of Freedom

Let dynamics of a conservative system be described by an equation of a more general form

$$D_0(p)x = 0, \tag{2.70}$$

where $D_0(p)$ is a dynamic stiffness operator of the system reduced to the impact point, $\operatorname{Im} D_0(ki\omega) \equiv 0$.

Imposing the impact conditions (2.47) on the solution of Eq. (2.70) and treating the motion as the system's response to a periodic sequence of impulses, we get

$$x(t) = -J\chi_1(t), \tag{2.71}$$

where J is still an impact impulse, $\chi_1(t)$ is a periodic Green's function of the first kind corresponding to the system (2.70)

$$\chi_1(t) = \frac{1}{T} + \frac{2}{T}\sum_{k=1}^{\infty} l_0(ki\omega)\cos k\omega t\,,$$
$$l_0(ki\omega) = D_0^{-1}(ki\omega), \quad \operatorname{Im} l_0(ki\omega) = 0. \tag{2.72}$$

An oscillation period and impact impulse are linked by expressions

$$x(0) = -J\chi_1(0) = \Delta\,, \quad J = -\Delta[\chi_1(0)]^{-1} \tag{2.73}$$

For a system with double-sided symmetric limiters we get analogously

$$x(t) = -J\chi_2(t), \tag{2.74}$$

where $\chi_2(t)$ is a periodic Green's function of the second kind corresponding to the system (2.70)

$$\chi_2(t) = \frac{4}{T}\sum_{k=1}^{\infty} l_0[(2k+1)i\omega]\cos(2k+1)\omega t\,. \tag{2.75}$$

An impact impulse is determined by an initial energy reserve in the system; an oscillation period is linked with the impulse by relations of the type (2.57)

$$x(0) = -J\chi_2(0) = \Delta\,, \quad J = -\Delta[\chi_2(0)]^{-1} \tag{2.76}$$

Thus, a one-impact T-periodic regime with parameters determined by Eqs. (2.73) or (2.76) occurs in the conservative vibroimpact system.

If dissipation during the motion and impact is low, then a regime close to resonance can exist in the system. In such a case the periodic oscillations are supported by a small external periodic force.

Existence conditions for the regime close to a resonance were studied in [85]. They are reduced to the demand of the energy balance for external and dissipative forces during the period of the generating solution of the conservative system. A method of construction of quasi-resonant solutions for a vibroimpact system with respect to the energy balance condition is used in [16].

Below, in Section 2.4, another scheme of successive approximations is given for a construction of quasi-resonant solutions for vibroimpact systems.

2.3 Optimal Periodic Control for Vibroimpact System, Linear between Impacts

We now consider in details the optimal control problem for a system with a one-sided limiter. Dynamics of the system is described by the equation

$$D(p)x = Q(p)u \tag{2.77}$$

and by impact conditions

$$x = \Delta\,, \qquad \dot{x}_+ = -R\dot{x}_-\,. \tag{2.78}$$

Here x is a relative coordinate of a striking element (Fig. 2.3), $D(p)$ is a dynamic stiffness operator of the system, $Q(p)$ is a transfer function of the regulator, $u(t)$ is a T-periodic scalar control. The form of the function $D(p)$ is determined by the system structure; parameters of the regulator are arbitrary in a general case, but it is supposed that the closed system with a transfer function $H(p) = O(p)D^{-1}(p)$ is stable, and $H(p) = O(p^{-2})$, $p \to \infty$.

2.3.1 Control for the Fixed Oscillation Period

Let us find a control u forming the maximum impact impulse

$$J = M(1+R)\dot{x}_- \tag{2.79}$$

of a one-impact T-periodic regime with the restriction

$$\Phi_1(u) = \frac{1}{T}\int_0^T u^2(t)dt \le C\,, \tag{2.80}$$

which determines energy costs of the control. Let us reduce the problem's demands to one functional

$$\Phi(u) = J - \frac{\alpha}{2T}\int_0^T u^2(t)dt, \quad \alpha > 0, \tag{2.81}$$

and determine the T-periodic control $u(t)$ from the maximum condition for the functional (2.81) corresponding to the one-impact T-periodic regime. If the period T is not fixed then it should be additionally determined from the optimality conditions.

Let us write an integral equation of the T-periodic regime for the system with a one-sided limiter

$$x(t) = -J\chi_1(t) + \int_0^T \chi_1^u(t-s)u(s)ds, \quad 0 < t < T. \tag{2.82}$$

Here

$$\chi_1(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} D^{-1}(ki\omega)e^{ki\omega t} \tag{2.83}$$

is a periodic Green's function of the disconnected system,

$$\chi_1^u(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} D^{-1}(ki\omega)Q(ki\omega)e^{ki\omega t} \tag{2.84}$$

is a periodic Green's function of the controlled system. Excluding the impulse J from (2.82), we get

$$\dot{x}_-(0) = [M(1+R)]^{-1}J = -J\dot{\chi}_{1-}(0) + \int_0^T [\chi_1^u(t-s)]_{t=0-} u(s)ds. \tag{2.85}$$

Accounting for obvious substitutions,

$$\chi_{1t}(t-s) = -\chi_{1s}(t-s), \quad \chi_1(t-s)\big|_{t=0-} = \chi_1(T-s),$$
$$[\chi_{1t}(t-s)]_{t=0-} = -\chi_{1s}(T-s), \tag{2.86}$$

we will reduce (2.82), (2.85) to the form

$$J = -\mu_1^{-1}\int_0^T \chi_{1s}^u(T-s)u(s)ds, \tag{2.87}$$

and

$$x(t) = \int_0^T K_1^u(t,s)u(s)ds, \tag{2.88}$$

where

$$x(t) = [(1+R)M]^{-1} + \dot{\chi}_{1-}(0) \tag{2.89}$$

and

$$K_1^u(t,s) = \chi_1^u(t-s) + \mu_1^{-1}\chi_1(t)\chi_{1s}^u(T-s). \tag{2.90}$$

Substituting (2.87) into (2.81), we will exclude J from the functional Φ

$$\Phi(u) = -\int_0^T \left[\mu_1^{-1}\chi_{1s}^u(T-s)u(s) + \frac{\alpha}{2T}u^2(s) \right] ds. \tag{2.91}$$

Thus, the problem is reduced to determination of the control $u(t)$ maximizing the functional (2.91) under an additional constraint

$$x(T) = x(t) = \Delta. \tag{2.92}$$

The condition of velocity discontinuity of (2.78) is accounted for in transformations (2.79), (2.85) and in the form (2.88) of motion equations. Problem restrictions should also contain demands

$$J \geq 0, \quad x(t) \leq \Delta, \tag{2.93}$$

determining a positive direction of the axis and excluding the overcoming of the limiter position. Conditions (2.93) are not included into the functional of the problem, and should be proved after its solution.

Thus, the problem is reduced to a control construction maximizing the functional (2.91) with constraints (2.88), (2.92). It is easy seen that condition (2.82) reduces the examined problem to a position control problem which is studied in details in Section 2.5. Indeed, owing to (2.88), condition (2.92) is reduced to an isoperimeter constraint

$$\int_0^T \left[K_1^u(T,s)u(s) - \Delta T^{-1} \right] ds = 0. \tag{2.94}$$

Substituting (2.94) into (2.91), we will write the extended functional of the problem in the form:

$$S = \int_0^T \left\{ -\left[\frac{\alpha}{2T}u^2(s) + \mu_1^{-1}\chi_s^u(T-s)u(s) \right] + \lambda\left[K_1^u(T,s)u(s) - \frac{\Delta}{T} \right] \right\} ds =$$

$$= \int_0^T L(s, u(s), \lambda)\, ds. \tag{2.95}$$

A condition of the maximum principle

$$\partial L / \partial u = 0 \tag{2.96}$$

and Eq. (2.94) serve for determination of the optimal control $u_*(s)$ and the multiplier λ. From (2.95), (2.96) we have

$$u_*(s) = \alpha_1^{-1}\left[\lambda\chi_1^u(T-s) + \mu_1^{-1}\chi_{1s}^u(T-s)(\lambda\chi(0)-1) \right], \tag{2.97}$$

$$\alpha_1 = \alpha/T .$$

Substituting (2.97) into (2.87), (2.94), we get

$$x(0) = -J\chi_1(0) + \alpha_1^{-1}\lambda k_{11} = \Delta ,$$
$$J = \alpha_1^{-1}\mu_1^{-2}\left[1-\lambda\chi_1(0)\right]k_{12} , \tag{2.98}$$

where

$$k_{11} = \int_0^T \left[\chi_1''(s)\right]^2 ds , \quad k_{12} = \int_0^T \left[\chi_{1s}''(s)\right]^2 ds . \tag{2.99}$$

Solving the system (2.99) with respect to J, λ, we get

$$J = J_* = k_{12}\left[k_{11}\alpha_1^{-1} - \Delta\chi_1(0)\right]D^{-1} ,$$
$$\lambda = \lambda_* = \left[k_{12}\chi_1(0) + \alpha_1\mu_1^2\Delta\right]D^{-1} \tag{2.100}$$

and

$$1-\lambda_*\chi_1(0) = \mu_1^2\left[k_{11} - \alpha_1\Delta\chi_1(0)\right]D^{-1} , \tag{2.101}$$

where

$$D = k_{11}\mu_1^2 + k_{12}\chi_1^2(0) . \tag{2.102}$$

The constraint (2.97) can be proved for concrete values of $\chi_1(T)$, μ_1, and is fulfilled, in particular, for $\mu_1 << 1$. A coefficient α should be chosen according to condition (2.80). Accounting for (2.97)–(2.101),

$$\frac{1}{T}\int_0^T u^2(t)dt = \frac{T}{\alpha^2}\left\{\lambda^2 k_{11} + \mu_1^2\left[k_{11} - \alpha_1\Delta\chi_1(0)\right]^2 D^{-2}k_{12}\right\} \le C . \tag{2.103}$$

In particular, for $\mu = 0$ we have $\lambda = \chi_1^{-1}(0)$ and

$$\alpha^2 \ge Tk_{11}C^{-1}\chi_1^{-2}(0) . \tag{2.104}$$

From the last condition we get the following restriction:

$$J_* = k_{11}\alpha^{-1}T - \Delta\chi_1^{-1}(0) \le \sqrt{k_{11}TC}\left|\chi_1(0)\right| + J_0 , \tag{2.105}$$

where $J_0 = -\Delta\chi_1^{-1}(0)$ is the impulse value at vibroimpact resonance. The inequality (2.105) allows us to estimate the ultimate possibilities of the system for limited energy costs of the control.

2.3.2 The Choice of Optimal Period between Impacts

Above it was supposed that the period T of guided excitation was fixed. It is a case when the energy supply is an independent energy source. At the same time, the oscillation period can be treated as a problem's parameter and as thus it can be chosen from optimality conditions (now not discussing a problem of practical realization). Then the condition (1.79)

$$\partial S/\partial T = 0 \tag{2.106}$$

is added to the optimality conditions. Here the functional S is determined by Eq. (2.95), and owing to (2.94), (2.96)

$$\frac{\partial S}{\partial T} = -\frac{\alpha}{2T^2}\int_0^T u^2(s)ds + \frac{\alpha}{2T}u^2(T). \tag{2.107}$$

Substituting (2.97), (2.103) into (2.107), (2.106), we get

$$\frac{\partial S}{\partial T} = \frac{T}{\alpha} - \left\{\left[\lambda\chi_1^u(0) - \mu_1\left(k_{11} - \frac{\alpha}{T}\Delta\chi(0)\right)D^{-1}\chi_{1+}^u(0)\right]^2 - \right.$$
$$\left. -\frac{1}{T}\left\{\lambda^2 k_{11} + \mu_1^2\left[k_{11} - \alpha_1\Delta\chi_1(0)\right]^2 D^{-2}k_{12}\right\}\right\} = 0. \tag{2.108}$$

The constants k_{11}, k_{12}, λ are determined by Eqs. (2.99), (2.100), and, in their turn, depend on T together with χ_1, χ_1^u. Generally speaking, Eq. (2.108) can not be solved analytically with respect to T. If dissipation in the system is low, $\mu \approx 0$ and $Q(p) = 1$, then $\chi_1^u(T) = \chi_1(T)$, $\lambda = \chi_1^{-1}(0)$, and the control (2.108) considerably simplifies:

$$\frac{\partial S}{\partial T} = \frac{T}{\alpha}\left[1 - \frac{1}{T\chi_1^2(0)}\int_0^T \chi_1^2(s)\right]ds = 0. \tag{2.109}$$

In particular, for a system with one degree of freedom Eq. (2.109) is reduced to the form

$$1 - \frac{1 + \sin\Omega T/\Omega T}{2\cos^2(\Omega T/2)} = 0, \tag{2.110}$$

and the optimal period of oscillations T_* can be found as the smallest root of the equation

$$x = \tan x, \qquad x_* = \Omega T_*, \tag{2.111}$$

or,

$$\Omega T_* \approx \frac{3\pi}{2} - \frac{2\pi}{3}. \tag{2.112}$$

In many of impulse systems the efficiency is increased not by means of the increase in the impact impulse, but either by increasing the impact frequency [16] or by decreasing the period between impacts.

Let dynamics of the system be described by Eq. (2.88) and condition (2.92). Let us find a control $u(t)$ minimizing the high-speed-action functional

$$\Phi(u)=\int_0^T dt \tag{2.113}$$

for a condition $|u|\leq U_0$. The Lagrange functional S has the form

$$S=\int_0^T\left\{-1+\lambda\left[K_1^u(T,s)u(s)-\Delta T^{-1}\right]\right\}ds=\int_0^T L(s,u,\lambda)ds, \tag{2.114}$$

where $K_1^u(T,s)$ is a kernel of (2.90), λ is a Lagrange multiplier. From the maximum condition for L with respect to u follows

$$u_*=U_0\operatorname{sgn}\left[\lambda K_1^u(T,s)\right]=U_0\operatorname{sgn}\lambda\operatorname{sgn}K_1^u(T,s). \tag{2.115}$$

Two conditions (2.92) and (2.106) serve for determination of λ, T. Owing to (2.114), (2.115), the Lagrange functional

$$S=-.T-\lambda\Delta+|\lambda|U_0q(T),$$
$$q(T)=\int_0^T\left|K_1^u(T,s)\right|ds>0, \tag{2.116}$$

and from conditions (2.92), (2.106) with account for (2.115), (2.116) follows

$$x(0)=x(T)=U_0\operatorname{sgn}\lambda q(t)=\Delta, \tag{2.117}$$
$$\frac{\partial S}{\partial T}=-1+|\lambda|U_0\frac{\partial q}{\partial T}.$$

From the first condition not only $\operatorname{sgn}\lambda=\operatorname{sgn}\Delta$ can be obtained, but also an equation for determination of the period T

$$q(T)=U_0^{-1}|\Delta|. \tag{2.118}$$

The second condition of (2.117) determines $|\lambda|$. (For calculation of the derivative dq/dt, a dependence of $K(t,s)$ on T should be accounted).

2.3.3 Determination of Optimal Clearance (Press Fit)

If a period of an external excitation is fixed, then an impact impulse can be in creased by means of optimal choice of a clearance (press fit). This problem – as other ones of the impact system optimization – was solved with the use of the maximum principle [15, 16, 67], but an analysis of differential equations made the solution for a system with only one degree of freedom possible.

Let us find an optimal control and an optimal clearance which form the maximum impulse in the system (2.77) with a one-sided limiter when the control $u(t)$ satisfies the restriction $|u(t)| \le U_0$.

Accounting for (2.87), let us write the functional

$$\Phi(u) = J = -\mu_1^{-1} \int_0^T \chi^{u}_{1s}(T-s)u(s)ds. \tag{2.119}$$

Including the constraint (2.94) into the functional (3.119), we get an extended functional analogous to (2.95):

$$S = \int_0^T \left\{-\mu_1^{-1}\chi^{u}_{1s}(T-s)u(s) + \lambda\left[K_1^{u}(T,s)u(s) - \Delta T^{-1}\right]\right\}ds = \int_0^T L(s,u(s),\lambda,\Delta)ds. \tag{2.120}$$

An additional condition (1.76) serves for determination of the parameter Δ. As far as Δ is not within integration bounds, then the condition (1.76) is reduced to the level (1.77), in our case

$$\int_0^T \frac{\partial L}{\partial \Delta} ds = 0, \tag{2.121}$$

that gives, owing to (2.120),

$$\lambda = 0. \tag{2.122}$$

Then the functionals S and Φ coincide, and from the maximum condition for L with respect to u we get, owing to (2.119), (2.120),

$$u_* = -U_0 \operatorname{sgn} \chi^{u}_{1s}(T-s), \tag{2.123}$$

i.e.,

$$J_* = \mu_1^{-1} U_0 \int_0^T |\chi^{u}_{1s}(s)| ds \tag{2.124}$$

and

$$\Delta_* = -J_* \chi_1(0) - U_0 \int_0^T \chi_1^{u}(T-s) \operatorname{sgn} \chi^{u}_{1s}(T-s)ds.$$

In particular, for a system with one degree of freedom without dissipative elements

$Q(p) = 1$, $\chi_1^{u}(t) = \chi_1(t)$, $\mu_1 = (1-R)[2(1+R)]^{-1}$

$$\chi_1(t) = \frac{1}{2\Omega}\frac{\cos\Omega(t - T/2)}{\sin \Omega T/2}, \quad \dot{\chi}_{1-}(0) = -\frac{1}{2},$$

and

$$u_*(s) = -U_0 \operatorname{sgn}\left[\sin \Omega(T/2 - s)\right], \quad J_* = U_0(\mu_1 \Omega)^{-1} \tan\frac{\pi\Omega}{2\omega},$$

that with accuracy up to notations coincides with results obtained in [67]. If the period T is also determined by the optimality conditions then Eq. (2.106) should be added to Eq. (2.121).

2.3.4 Optimal Control in Systems with Double-Sided Symmetric Limiters

For an optimal control construction in systems with double-sided limiters a procedure of the method of integral equations can be also used. So, only one model is treated: the optimal control construction forming a regime optimal with respect to a high-speed action in the system (2.77) with double-sided limiters and control restrictions.

The system dynamics is still described by Eq. (2.77)

$$D(p)x = Q(p)u,$$

where $u(t)$ is a T-periodic control, with a property

$$u(t) = -u(t + T/2). \tag{2.125}$$

Impact conditions are written with account for the choice of the axis direction

$$\begin{gathered} t = 0, \quad x = \Delta, \quad \dot{x}_+ = -R\dot{x}_-, \quad \dot{x}_- > 0, \\ t = T/2, \quad x = -\Delta, \quad \dot{x}_+ = -R\dot{x}_-, \quad \dot{x}_- < 0. \end{gathered} \tag{2.126}$$

When conditions (2.125), (2.126) are fulfilled, then a periodic motion has the property

$$x(t + T/2) = -x(t), \quad x(t + T) = x(t) \tag{2.127}$$

and can be treated on the interval $0 < t < T/2$. And

$$x(t) = -J\chi_2(t) - \int_0^{T/2} \chi_2^u(t - s)u(s)ds, \quad 0 < t < T/2, \tag{2.128}$$

where $\chi_2(t)$ and $\chi_2^u(t)$ are periodic Green's function of the second kind:

$$\begin{gathered} \chi_2(t) = \frac{2}{T}\sum_{k=-\infty}^{\infty} D^{-1}[(2k+1)i\omega]\exp[(2k+1)i\omega t], \\ \chi_2^u(t) = \frac{2}{T}\sum_{k=-\infty}^{\infty} D^{-1}[(2k+1)i\omega]M[(2k+1)i\omega]\exp[(2k+1)i\omega t]. \end{gathered} \tag{2.129}$$

The control $u(t)$ should be found from the minimum condition of the high-speed-action functional (2.113) with the restriction $|u| \le U_0$. Excluding the impulse J from (2.129) with the help of the velocity discontinuity condition (2.126)

$$J = -\mu_2^{-1} \int_0^{T/2} \chi_{2s}^{u}(T/2 - s) u(s) ds, \tag{2.130}$$

where

$$\mu_2 = [M(1+R)]^{-1} + \dot{\chi}_{2-}(0), \tag{2.131}$$

let us reduce (1.128) to the integral form

$$x(t) = \int_0^{T/2} K_2^{u}(t,s) u(s) ds, \tag{2.132}$$

where

$$K_2^{u}(t,s) = -\mu_2^{-1} \chi_{2s}^{u}(T/2 - s) \chi_2(t - T/2) + \chi_2^{u}(t-s). \tag{2.133}$$

Here the condition $\chi_2(t) \equiv -\chi_2(t - T/2)$ is accounted.

The Lagrange function of the problem under study is analogous to (2.114):

$$L(s,u,\lambda,T) = -1 + \lambda [K_2^{u}(0_+, s) u(s) - \Delta T^{-1}], \tag{2.134}$$

$$u_*(s) = \arg \max_{|u| \le U_0} L(s,u,\lambda,T) = -U_0 \operatorname{sgn} \lambda \operatorname{sgn} K_2^{u}(0_+, s), \tag{2.135}$$

and

$$K_2^{u}(0_+, s) = -K_2^{u}(T/2, s) = -[\mu_2^{-1} \chi_{2s}^{u}(T/2 - s) \chi_2(0) + \chi_2^{u}(T/2 - s)]. \tag{2.136}$$

Substituting (2.135) into (2.133) and imposing the impact condition on $x(t)$, we get

$$x(T/2) = -\Delta = -U_0 \operatorname{sgn} \lambda \int_0^{T/2} |K_2^{u}(T/2, s)| ds,$$

i.e., $\operatorname{sgn} \lambda = \operatorname{sgn} \Delta$, and the equation

$$U_0 \int_0^{T/2} |K_2^{u}(T/2, s)| ds = \Delta \tag{2.137}$$

can serve for an oscillation period determination.

Let us examine an example of an asymmetric motion of a system in analogy with Section 1.3.

Let a dynamic system (manipulator) move between the limiters which are symmetrically positioned at $x = \pm\Delta$. At impact against the limiter, the system's velocity is damped by a holder, a loading (unloading) and a respective change of dynamic characteristics of the system occur during the stop [17]. The system leaves the holder with a zero velocity, and the motion continues up to the impact against another limiter. If we exclude the stop from our consideration, then the system's motion can be schematically presented in a way shown in Fig. 2.9. The cycle duration (excluding the stop) is $T = (T_1 + T_2)/2$.

Let us construct a control $u^1(t)$, $u^2(t)$, $|u^j| \le U_0$, minimizing duration of each of the intervals $T_1/2$, $T_2/2$. Here, as in Section 1.3, each of the intervals is treated as the first half-period of respective asymmetric motion.

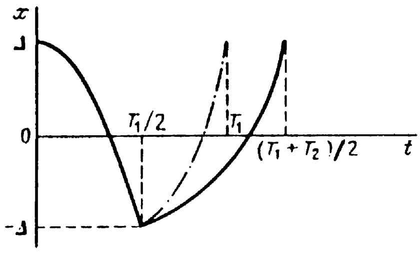


Fig. 2.9

Let the moment of the impact against the right limiter be the begin of the time axis. Then, at the interval $0 < t < T_1/2$ the system dynamics is described by the equation

$$D_1(p)x = Q(p)u^1(t) \tag{2.138}$$

and by the impact condition

$$t = 0, \quad x(0) = \Delta, \quad \dot{x}_-(0) > 0, \quad \dot{x}_+(0) = 0. \tag{2.139}$$

Here $D_1(p)$ is an operator of dynamic stiffness of the actuator, $Q(p)$ is a characteristic of the control network, independent of system parameters. The equality $\dot{x}_+(0) = 0$ means that the condition of velocity discontinuity can be written in the traditional form $\dot{x}_+ = -R\dot{x}_-$ for $R = 0$; a mass of the striking element is equal to m_1.

Just as for the interval $0 < t < T_2/2$, the system dynamics is described by the equation

$$D_2(p)x = Q(p)u^2(t) \tag{2.140}$$

and by the condition of the impact against the left limiter

$$t = 0, \quad x = -\Delta, \quad \dot{x}_+ = -R\dot{x}_-, \quad \dot{x}_- < 0, \quad R = 0, \tag{2.141}$$

a mass of the striking element is equal to m_2.

From (2.135)–(2.137) follows that an optimal control at each of the intervals is described by

$$u_*^j(s) = (-1)^j U_0 \operatorname{sgn} K_2^{ju}(T_j/2, s), \quad 0 < s < T_j/2, \tag{2.142}$$

where

$$K_2^{ju}(T_j/2,s) = -\left[\left(\mu_2^j\right)^{-1}\chi_{2s}^{ju}(T_j/2-s)\chi_2^j(0)+\chi_2^{ju}(T_j/2-s)\right],$$

and $\chi_2^j(s)$, $\chi_2^{ju}(s)$ are periodic Green's functions analogous to (2.129), $\mu_2^j = -m_j^{-1}+\dot{\chi}_{2-}^j(0)$, $j=1$, 2. A duration of the interval $T_j/2$ is determined by the condition (2.137), namely,

$$U_0\int_0^{T/2}\left|K_2^{ju}(T_j/2,s)\right|ds = |\Delta|. \tag{2.143}$$

Let the manipulator, for instance, be schematized by a system with one degree of freedom, and $Q(p)=1$, $m_1 = m_2 = 1$. Then

$$\chi_2^j(t) = \frac{1}{2\Omega_j}\frac{\sin\Omega_j(t-T_j/4)}{\cos\Omega_jT_j/4}, \qquad 0<t<T_j/2,$$

$$\mu_1 = \mu_2 = 1/2,$$

and

$$K_2^j(T_j/2,s) = -\frac{\sin\Omega_j(t-T_j/2)}{2\Omega_j\cos^2\Omega_jT_j/4},$$

so that

$$u_*^1(t) = U_0\operatorname{sgn}\sin\Omega_1(t-T_1/2),$$
$$u_*^2(t) = -U_0\operatorname{sgn}\sin\Omega_2(t-T_2/2).$$

Intervals T_1, T_2 are determined by relations

$$\frac{U_0}{2\Omega_j\cos^2(\omega T_j/4)}\int_0^{T_j/2}\left|\sin\Omega_j(t-T_j/2)\right|dt = \Delta, \qquad j=1,\ 2.$$

Suppose $\Omega_jT_j<2\pi$. Then $u^1=-U_0$, $u^2=U_0$, and the intervals T_1, T_2 are determined from the equation

$$\tan^2(\Omega_jT_j/4) = \Delta U_0^{-1}.$$

If the system has more than one degree of freedom, then motions of intermediate elements are connected in the same way as in Section 1.5.

2.4 Optimal Control for Quasi-Resonant Systems

It was shown in Section 2.2 that in a case of a small energy dissipation a resonant regime where the motion is with a finite impulse is supported by a small periodic excitation can be realized in the system. Obviously, a special attention at a choice of an optimal scheme should be paid to resonant systems while in such a case it is possible to achieve a considerable effect with the use of low control excitations. In this case the control compensates an effect of dissipative forces and realizes a periodic regime which is close to a resonant one. Its parameters are determined by optimality conditions. If the system contains nonlinear or non-conservative elements, or a control is not additive (for instance, in the form of parametric action), then approximate methods analogous to those analyzed in Section 1.6 can be used for an optimal control construction.

2.4.1 General Equations of the Method of Successive Approximations for Search of Periodic Solutions for Vibroimpact Systems

Let us limit our considerations by a case of a one-impact T-periodic motion of a system with one striking element and a one-sided limiter.

Considering the impact moment to be the begin of a time axis, we can write the motion equations of the system between impacts in the form

$$\dot{x} = Ax + g(x,u), \tag{2.144}$$

where $x \in R_n$, $u \in U \subset R_m$, A is a matrix of the respective dimension, ε is a small parameter. It is assumed that eigenvalues of the matrix A differs from $\pm i2\pi k/T$ ($k = 0, \pm 1, ...$). Demands to the function g are listed below.

Let x_1 be a coordinate of the striking element, $x_2 = \dot{x}_1$ be its velocity. Then the impact condition has the form

$$t = kT, \quad x_1 = \Delta, \quad x_{2+} = -Rx_{2-} \quad (k = 0, \pm 1, ...). \tag{2.145}$$

The piecewise continuous function $u(t)$ will be called "admissible control" if $u(t) \in U$, and for $u = u(t)$ there exists a unique T-periodic solution of the system (2.144), (2.145).

We will look for a control $u_*(t)$ from a set of admissible controls minimizing the functional

$$\Phi_\varepsilon(u) = \int_0^T [\theta(x) + \psi(u)] dt \tag{2.146}$$

on the T-periodic solution of the system (2.144), (2.145). Inequalities (3.93) are supposed to be fulfilled, and they are proved after the problem is solved.

The optimal control u_* is determined by the condition

$$u_* = \arg\max H(x,u,q)/u \in U, \tag{2.147}$$

where $H(x,u,q)=q'(Ax+\varepsilon g(x,u))-\theta(x)-\psi(u)$, q is the solution of the conjugate system (1.85)

$$\dot{q}=-\partial H/\partial x=-A'q-\varepsilon g_x'(x,u)q\,, \tag{2.148}$$

satisfying the periodicity conditions $q_j(t+T)=q_j(t)$ and the condition in the discontinuity point [see (A.24)]

$$t=0\,,\quad q_{2+}=-R^{-1}q_{2-}\,,\quad q_{j+}=q_{j-}\,,\quad j\neq 2\,, \tag{2.149}$$

where q_j is the j-th component of the vector q, $j=1$, ..., n.

Suppose that Eq. (2.147) is solvable with respect to u, and

$$u_*=V(x,q)\,. \tag{2.150}$$

Thus, for a solution of the periodic control optimization problem a periodic solution of a nonlinear system of equations

$$\begin{aligned}&\dot{x}=Ax+\varepsilon f_1(x,u)\,,\\&\dot{q}=-A'q+\varepsilon f_2(x,q)-\theta_x(x)\end{aligned} \tag{2.151}$$

with discontinuity conditions

$$t=0\,,\quad x_1=\Delta\,,\quad x_{2+}=-Rx_{2-}\,,\quad q_{2+}=-R^{-1}q_{2-}\,. \tag{2.152}$$

should be found. Here

$$f_1(x,q)=g(x,u)\,,\qquad f_2(x,q)=g_x'(x,u)$$

for $u=V(x,q)$.

Suppose that the right-hand parts of the equations satisfy the following conditions [84, 85]:

1) Functions f_1, f_2, θ are uniquely determined for x, $q\in G$, where G is some domain in R_{2n}.

2) The domain can be divided by discontinuity surfaces

$$v_k(x,q)=0 \tag{2.153}$$

into domains $G_1,\ldots,G_l$, in each of which the functions $f_{1,2}(x,q)$ are two times continuously differentiable with respect to x, q up to the boundaries, the function $\theta(x)$ is three times continuously differentiable. The functions $v_k(x,q)$ are two times continuously differentiable with respect to x, q for x, $q\in G$. (The straight line $x_1=\Delta$ is also included in the set of discontinuity surfaces.)

3) Either the functions f_1, f_2, θ or their first or second derivatives with res-

pect to x, q can have discontinuities of the first kind on the discontinuity surfaces.

4) It is further supposed that the generating system

$$\dot{x}^0 = Ax^0, \qquad \dot{q} = -A'q + \theta'_x(x^0) \tag{2.154}$$

has a unique T-periodic solution satisfying the conditions (2.152) (this solution is constructed in the same way as in Section 2.1).

Suppose that conditions

$$(v_{kx})'\dot{x} + (v_{kq})'\dot{q} \neq 0, \quad x = x^0, \quad q = q^0 \tag{2.155}$$

are fulfilled in the crossing points with the surfaces (2.154).

It was proved in [85] for more general assumptions about the form of the system (2.151) and discontinuity conditions (2.152) that for our assumptions it is possible to construct a solution for the system (2.151), (2.152) with the use of the method of a small parameter.

Theorem 2.1 [85]. *Let the right-hand parts of Eq.* (2.151) *and the generating solution* $x^0(t)$, $q^0(t)$ *satisfy the conditions* 1) – 4). *Then for sufficiently small* ε *the following scheme of successive approximations holds*:

$$\dot{x}^l = Ax^l + \varepsilon f_1(x^{l-1}, q^{l-1}), \qquad \dot{q}^l = -A'q^l + \varepsilon f_2(x^{l-1}, q^{l-1}) - \theta_x(x), \tag{2.156}$$

$$x^l(t) = x^l(t+T), \quad q^l(t) = q^l(t+T), \qquad t = 0, \quad x_1^l = \Delta, \quad x_{2+}^l = -Rx_{2-}^l, \quad q_{2+}^l = -R^{-1}q_{2-}^l, \tag{2.157}$$

and

$$\left|x^l(t) - x_*(t)\right| \le c\varepsilon^{l+1}, \quad \left|q^l(t) - q_*(t)\right| \le c\varepsilon^{l+1}, \qquad 0 < t < T, \quad l = 1, 2, \ldots \tag{2.158}$$

Below we will examine the quasi-conservative systems for which the recovery coefficient $R = 1 - \varepsilon r$. In order to find a generating solution, the relation $R = 1$ should be used, and in the subsequent approximations the relation $R = 1 - \varepsilon r$ should be accounted.

Let us return to the optimal control problem. Define the control

$$u^l(t) = V(t, x^l(t), q^l(t)). \tag{2.159}$$

It is also easy to show – as in Section 1.6 – that the following statement holds:

Theorem 2.2. *Let the functions* f_1, f_2, θ *satisfy the conditions of* Theorem 2.1, *the function* $\psi(u)$ *be continuous and continuously differentiable for* $u \in U$.

Then for sufficiently small ε

$$0 \le \Phi_\varepsilon\left(u^l\right) - \Phi_\varepsilon\left(u_*\right) \le c\varepsilon^{l+1},$$

$$\left|\int_0^T \left[u_*(s) - u^l(s)\right] ds\right| \le c\varepsilon^{l+1}, \tag{2.160}$$

$$\left|x_*(t) - x_*^l(t)\right| \le c\varepsilon^{l+1},$$

where u_* *is the optimal control with respective trajectory* x_*, x_*^l *is a solution of the system* (2.144) *for* $u = u^l$, $\Phi_\varepsilon\left(u^l\right)$ *is a value of the functional* (2.146) *for* $u = u^l$, $x = x^l$.

If the functions f_1, f_2 are continuous and satisfy the Lipschitz condition with respect to x, q on discontinuity surfaces, then the estimate with respect to u can be strengthened:

$$\left|u_*(t) - u^l(t)\right| \le c\varepsilon^{l+1}, \tag{2.161}$$

Finally, the scheme of successive approximations of Theorem 1.5 holds.

Theorem 2.3. *Let the periodic motion of the system be described by* Eqs. (2.144), (2.145), *and* $\tilde{u}^l$ *be a control minimizing the functional* (2.146) *on trajectories of the system*

$$\dot{\tilde{x}}^l = A\tilde{x}^l + \varepsilon g\left(t, \tilde{x}^{l-1}, u\right), \quad l = 1, \ldots, \tag{2.162}$$

where the generating approximation is determined by Eq. (2.154). *Then in a case of fulfillment of enumerated smoothness conditions, the estimates*

$$0 \le \Phi_\varepsilon\left(u^l\right) - \Phi_\varepsilon\left(u_*\right) \le c\varepsilon^{l+1},$$

$$\left|\tilde{x}^l(t) - x_*(t)\right| \le c\varepsilon^{l+1}, \tag{2.163}$$

are true. Here $x_*^l(t)$ *is a solution of the system* (2.144) *for* $u = \tilde{u}^l(t)$, $\Phi_\varepsilon\left(\tilde{u}^l\right)$ *is a value of the functional* (2.146) *for* $u = \tilde{u}^l$, $x = x_*^l$. *Respective estimates are also true for* $\tilde{u}^l$.

Thus, schemes of successive approximations which hold for systems described by differential equations can be applied to vibroimpact systems. Below we use integral equations of the vibroimpact system motion for periodic motion description. The equivalence of the optimality conditions and of the schemes of successive approximations for presentations in the form of differential and integral equations can be proved in the way used in Section 1, and is not separately discussed.

All the schemes of successive approximations can be also applied to the optimal high-speed-action problem. In this case the conclusions of Theorem 1.6 hold.

Let us note that the generating system (for $\varepsilon = 0$) is not controllable, but the generating trajectory x^0 remains in the ε-zone of the optimal trajectory. Moreover, for each admissible control $\tilde{u}(t)$, the solution of the system (1.244), (1.245) remains in the ε-zone of the optimal trajectory. Thus, a formation of the optimal control u^0 does not improve an ε-estimate of the closeness of trajectories in the first approximation, but it minimizes the control costs (analogous situations are discussed in [110]).

2.4.2
Optimal Control of Quasi-Resonant Motions of Vibroimpact Systems

Let us again examine the system with one striking element and with the one-sided limiter.

Let the system in periods between impacts be described by the equation

$$\left[D_0(p) + \varepsilon D_1(p)\right]x + \varepsilon g(x, \dot{x}) = \varepsilon Q(p)u\,, \tag{2.164}$$

where x is a coordinate of the striking element. Conditions of the impact against the one-sided limiter have the form

$$t = 0\,, \qquad x = \Delta\,, \qquad \dot{x}_+ = -R\dot{x}_-\,, \qquad R = 1 - \varepsilon r\,. \tag{2.165}$$

Here $g(x, \dot{x})$ is a reaction of an additional nonlinear link, $D_0(p)$ and $D_1(p)$ are conservative and dissipative parts of a dynamic stiffness operator for the linear part of the system, $Q(p)$ is a characteristic of the regulator for the coordinate x, ε is a small parameter. For a sake of simplicity, we consider the mass of the striking element to be equal unity, $g = g(x)$.

An equation of the quasi-resonant periodic motion get the form

$$x(t) = -J\chi_1^0(t) + \varepsilon\int_0^T \left\{\chi_1^u(t-s)u(s) - \chi_1^0(t-s)g\left[x(s)\right] - \chi_1^1(t-s)x(s)\right\}ds\,, \tag{2.166}$$

where $\chi_1^0(t)$, $\chi_1^1(t)$ and $\chi_1^u(t)$ are periodic Green's function of the first kind for respective elements of the system:

$$\begin{aligned}
\chi_1^0(t) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} D_0^{-1}(ki\omega)e^{ki\omega t}\,, \qquad D_0(ki\omega) \neq 0\,,\\
\chi_1^1(t) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} D_0^{-1}(ki\omega)D_1(ki\omega)e^{ki\omega t}\,, \qquad D_1^{-1}(ki\omega) \neq 0\,,\\
\chi_1^u(t) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} D_0^{-1}(ki\omega)Q(ki\omega)e^{ki\omega t}\,, \qquad Q^{-1}(ki\omega) \neq 0\,,
\end{aligned} \tag{2.167}$$

and it is supposed that $D_0^{-1}(p) = O\left(p^{-2}\right)$, $Q(p)D_0^{-1}(p) = O\left(p^{-2}\right)$, $D_1(p)D_0^{-1}(p)$

$= O\left(p^{-1}\right)$. In an approximate solution it is expedient to single out the generating system by means of calculating the impulse J from the condition $x(0) = \Delta$. We get

$$J = J_0 + \varepsilon\left[\chi_1^0(0)\right]^{-1}\int_0^T \left\{\chi_1^u(T-s)u(s) - \chi_1^0(T-s)g[x(s)] - \chi_1^1(T-s)x(s)\right\}ds, \tag{2.168}$$

where

$$J_0 = -\Delta\left[\chi_1^0(0)\right]^{-1} \tag{2.169}$$

is the impulse value at the vibroimpact resonance. From (2.166), (2.168) follows

$$x(t) = -J_0\chi_1^0(t) + \varepsilon\int_0^T \left\{Q_1^u(t,s)u(s) - Q_1^0(t,s)g[x(s)] - Q_1^1(t,s)x(s)\right\}ds, \tag{2.170}$$

where

$$\begin{aligned} Q_1^0(t,s) &= \chi_1^0(t-s) - \left[\chi_1^0(0)\right]^{-1}\chi_1^0(t)\chi_1^0(T-s), \\ Q_1^u(t,s) &= \chi_1^u(t-s) - \left[\chi_1^0(0)\right]^{-1}\chi_1^0(t)\chi_1^u(T-s), \\ Q_1^1(t,s) &= \chi_1^1(t-s) - \left[\chi_1^0(0)\right]^{-1}\chi_1^0(t)\chi_1^1(T-s). \end{aligned} \tag{2.171}$$

The generating system is not controllable for $\varepsilon = 0$, $x^0(t) = -J\chi_1^0(t)$, and, in the first approximation, the impact impulse $J = J_0$ is entirely determined by the values of Δ and T. So, for the fixed values of Δ and T, choosing the control, it is possible to achieve only small – of the order of ε – increase in the impact impulse. Thus, if the aim is the impulse increase then at least one of parameters (Δ or T) should be chosen from the optimality conditions. If the aim is a realization of the resonant regime then the control $u(t)$ should only support the generating regime compensating the energy losses in the system.

In the general case the problem's functional has the form

$$\Phi(u) = \int_0^T \left[\theta(x) + \psi(u)\right]dt, \quad u \in U, \tag{2.172}$$

the motion equations (2.170) and impact conditions (2.165) serve as additional constraints. Excluding the impulse J from the velocity discontinuity condition

$$J = -\frac{4}{r}\int_0^T \left\{\chi_{1s}^u(T-s)u(s) - \chi_{1s}^0(T-s)g[x(s)] - \chi_{1s}^1(T-s)x(s)\right\}ds, \tag{2.173}$$

and substituting (2.173) into (2.166), we get

$$x(t) = \int_0^T \left\{\left[K_1^u(t,s) + \varepsilon\chi_1^u(t-s)\right]u(s) - \left[K_1^0(t,s) + \varepsilon\chi_1^0(t-s)\right]g[x(s)] - \right.$$

$$-\left[K_1^1(t,s)+\varepsilon\chi_1^1(t-s)\right]x(s)\Big\}ds\,. \tag{2.174}$$

where

$$\begin{aligned}
K_1^0(t,s)&=\frac{4}{r}\chi_{1s}^0(T-s)\chi_1^0(t),\\
K_1^u(t,s)&=\frac{4}{r}\chi_{1s}^u(T-s)\chi_1^0(t),\\
K_1^1(t,s)&=\frac{4}{r}\chi_{1s}^1(T-s)\chi_1^0(t).
\end{aligned} \tag{2.175}$$

The function $\chi_{1s}^1(T-s)$ can contain a singular component of the kind of the δ^T – function, thus the equalities (2.173)–(2.175) can be understood in a generalized sense.

Accounting for the equality $x(0)=x(T)=\Delta$, let us replace the impact condition with an isoperimeter relation following from (2.174):

$$\int_0^T\Big\{\left[K_1^u(T,s)+\varepsilon\chi_1^u(T-s)\right]u(s)-\left[K_1^0(T,s)+\varepsilon\chi_1^0(T-s)\right]g[x(s)]-$$

$$-\left[K_1^1(T,s)+\varepsilon\chi_1^1(T-s)\right]x(s)-\Delta T^{-1}\Big\}ds=0\,. \tag{2.176}$$

Thus, the problem is reduced to a construction of the control $u(t)$ minimizing the functional (2.152) on the trajectory of the system (2.170) with the additional isoperimeter constraint (2.176).

Let us construct according to Theorem 2.3 an approximate scheme for determination of the optimal control $u_*(t)$ and respective optimal trajectory $x_*(t)$.

Let us find a control $u^0(t)$ minimizing the functional (2.172) on the trajectories of the generating system

$$x(t)=x^0(t)=-J_0\chi_1^0(t),\qquad J_0=-\Delta\left[\chi_1^0(0)\right]^{-1} \tag{2.177}$$

with the isoperimeter condition

$$\int_0^T K_1^u(T,s)u(s)-K_1^0(T,s)g\left[x^0(s)\right]ds=\int_0^T K_1^1(T,s)x^0(s)ds+\Delta\,. \tag{2.178}$$

After obvious transformations, condition (2.178) can be reduced to the following form:

$$\int_0^T\left\{\chi_{1s}^u(T-s)u(s)-\chi_{1s}^0(T-s)g\left[x^0(s)\right]\right\}ds=-(\gamma_1+r/4)J_0\,, \tag{2.179}$$

where

$$\gamma_1 = \int_0^T \chi_{1s}^1(T-s)\chi_1^0(s)ds = -\frac{1}{T}\sum_{k=-\infty}^{\infty}(ki\omega)\left|D_0(ki\omega)\right|^{-2} D_1(ki\omega). \tag{2.180}$$

It can be shown [16] that γ_1 is a reduced work of dissipative forces for the generating solution (2.177).

If $g(x)$ is a conservative force, $g(x^0) = g(-J\chi_1^0)$, then, obviously,

$$\int_0^T \chi_{1s}^0(T-s)g\left[-J\chi_1^0(s)\right]ds \equiv 0,$$

and Eq. (2.149) can be reduced to the form

$$\int_0^T \chi_{1s}^u(T-s)u(s)ds = -J_0\mu_1. \tag{2.181}$$

where

$$\mu_1 = \gamma_1 + r/4 \geq 0 \tag{2.182}$$

is a coefficient characterizing dissipative forces in the system and at impact.

The Lagrange function L^0 of the generating problem has the form

$$L^0(u,\lambda,T) = -\theta\left(x^0(s)\right) - \psi\left(u(s)\right) + \lambda\left\{\chi_{1s}^u(T-s)u(s) + J_0T^{-1}\mu_1\right\}. \tag{2.183}$$

In other words, the function L^0 does not depend on the unknown trajectory.

Working in the same way as in Section 1.6, we get that the optimal control

$$u^0(s) = \arg\max L^0(u,\lambda,T)/u \in U,$$

can be expressed by the function of the form

$$u^0(s) = U\left[y(s)\right] = V(s,\lambda), \tag{2.184}$$

where

$$y(s) = \lambda\chi_{1s}^u(T-s). \tag{2.185}$$

Substituting (2.184) into (2.181), we get the equation linking λ, Δ and T. If parameters Δ and T are unknown then additional conditions (2.106), (2.121) serve for their determination.

The scheme directly following from Theorem 2.3 was proposed for a construction of successive approximations [75].

Let us for the sake of simplicity suppose that the only discontinuity surface is the straight line $x = \Delta$. Obviously, in such a case the generating approximation (2.177) satisfies the condition (2.155). Let further the function $g(x)$ be three times continuously differentiable for all x, the function $V(s,\lambda)$ be piecewise continuous with respect to s, λ, and there exists a unique isolated solution of the generating problem.

The following scheme of successive approximations is constructed: we search for a control $\tilde{u}^l(s)$ minimizing the functional (2.172) on the trajectories of the system

$$\tilde{x}^l(t) = -J_0\chi_1^0(t) + \varepsilon\int_0^T \left\{\chi_1^u(t-s)\tilde{u}^l(s) - \chi_1^0(t-s)g\left[\tilde{x}^{l-1}(s)\right] - \right.$$
$$\left. -\chi_1^1(t-s)\tilde{x}^{l-1}(s)\right\}ds, \tag{2.186}$$

$$J^l = (1+R)\dot{\tilde{x}}_-^l(0), \quad \tilde{x}^l(0) = \Delta, \quad R = 1 - \varepsilon r,$$

where $\tilde{x}^{l-1}(t)$ is a solution of the previous optimal problem. Then there exists such a value ε_0, that for $0 \le \varepsilon < \varepsilon_0$ the estimates

$$\left|\int_0^T \left[u_*(t,s) - u^l(t,\varepsilon)\right]dt\right| \le c\varepsilon^{l+1}$$
$$\left|x_*(t) - \tilde{x}^l(t)\right| \le c\varepsilon^{l+1}, \tag{2.187}$$
$$\left|J_* - J^l\right| \le c\varepsilon^{l+1}, \quad c = \text{const}, \quad 0 < t < T,$$

hold, where u_*, x_*, T_* are solutions of the optimal problem. If the values Δ or T are also determined from the optimality conditions then they are approximated at each iteration stage with the accuracy $O\left(\varepsilon^{l+1}\right)$.

Let further $x_*^l(t)$ be a solution of the system (2.164), (2.165) for $u = u^l(t)$. Then

$$\left|\tilde{x}_*^l(t) - x_*(t)\right| \le c\varepsilon^{l+1}, \tag{2.188}$$

and parameters J, Δ, T and a value of the functional (2.164) are also approximated with the accuracy $O\left(\varepsilon^{l+1}\right)$.

The estimates (2.187), (2.188) are also true in the case of $g = g(x,z)$, $z = L_1(p)x$. Then for each step it should be considered

$$g^l(s) = g\left(x^l(s), z^l(s)\right), \quad z^l(s) = L_1(p)x^l(s). \tag{2.189}$$

The geometric restrictions (2.93) should be additionally controlled. We can directly make sure that these conditions are fulfilled in a sufficiently small ε – zone of the generating solution (2.177).

We will examine below the systems both with one-sided and double-sided impacts. The approximating scheme of the Theorem holds also for them.

Let us examine some examples.

1) Let $\psi(u) = \frac{\alpha}{2} u^2$, and there are no additional constraints. Then

$$u^0(s) = \alpha^{-1} \chi_{1s}^u (T - s); \tag{2.190}$$

the condition (2.181)

$$\lambda \alpha^{-1} k_{12}^u = -J_0 \mu_1$$

serves for determination of the parameter λ; here

$$k_{12} = \int_0^T \left[\chi_{1s}^u(s)\right]^2 ds, \tag{2.191}$$

i.e.,

$$u^0(s) = -J_0 \chi_{1s}^u (T - s) \mu_1 (k_{12})^{-1}; \quad J_0 = -\Delta\left[\chi_0(0)\right]^{-1}. \tag{2.192}$$

The control (2.192) is a control with minimal energy costs supporting a resonant regime; in the first approximation

$$V = \frac{1}{T} \int_0^T u^2(s) ds = \mu_1^2 J_0^2 T^{-1}.$$

In contrast to linear systems, a mean-square-bounded control realizes a periodic regime also in the absence of periodic excitation.

2) Let $\psi(u) = 0$, and the control constrains are determined by the inequality $|u| \le U_0$. Then

$$u^0(s) = -U_0 \operatorname{sgn}\left[\lambda \chi_{1s}^u (T - s) \chi_1^0(s)\right]; \tag{2.193}$$

The condition (2.181) can not determine the constant λ and should serve for the choice of Δ or T. Suppose for the sake of simplicity that one of the problem's parameters, Δ or T, is fixed; then from (2.181) follows that $\operatorname{sgn} \lambda = \operatorname{sgn} \Delta$ and the equations

$$\begin{gathered} u^0(s) = -U_0 \operatorname{sgn} \chi_{1s}^u (T - s), \\ |\Delta| = U_0 \mu_1^{-1} \left|\chi_1^0(0)\right| \int_0^T \left|\chi_{1s}^u(s)\right| ds \end{gathered} \tag{2.194}$$

serve for the determination of the optimal control $u^0(s)$ and of the unknown parameter Δ or T; the coefficient λ can be found from conditions (2.121), (2.106).

2.4.3
Choice of Minimal Duration of Working Cycle

The control of quasi-resonant oscillations is reduced to a resonant regime support; an impulse increase by means of the control is not considerable. Thus, the productivity growth can be achieved by increase of the number of impacts in a time unit, i.e., reducing a duration T of the working cycle. At the same time, the impulse $J_0 = -\Delta\chi_0^{-1}(0)$ depends directly on T, so at reduction of T it is necessary to obtain the impulse increase.

Let us find a control $u(t)$, $|u| \le U_0$, realizing a regime with the maximal impulse and minimal duration of the working cycle in the system (2.164), (2.162). The functional of the problem has the form

$$\Phi(u) = \beta J - \int_0^T dt\,. \tag{2.195}$$

Limiting our consideration to the case of the system, linear between impacts, let us write the Lagrange function of the generating problem. According to (2.183),

$$L^0 = \beta J_0 - 1 + \lambda\left\{\chi_{1s}^u(T-s)u(s) + \mu J_0 T^{-1}\right\} = L^0(u,\lambda,T)\,, \tag{2.196}$$

where J_0, x^0 are determined by Eq. (2.177) and depend directly on T. The optimal control $u^0(s)$ and the optimal oscillation period are expressed by (2.193), (2.194).

In particular, in a system with one degree of freedom for $Q(p) = 1$

$$x_1^u(t) = \chi_1^0(t) = \frac{1}{2\Omega}\frac{\cos\Omega(t - T/2)}{\sin\Omega T/2} = \chi_1^0(T-t)\,,$$

$$\chi_{1u}^u(T-t) = \chi_{1t}^u(t) = \frac{1}{2\Omega}\frac{\sin\Omega(T/2 - t)}{\sin\Omega T/2}\,.$$

We should bear in mind that the control supports a resonant regime in the system, and the oscillation period $T = 2\pi/\omega$ should satisfy the conditions

$$\Omega T/2 = \pi\Omega/\omega < \pi/2\,, \qquad \Delta < 0\,;$$

$$\pi/2 < \Omega T/2 < \pi\,, \qquad \Delta > 0\,,$$

i.e., it is always $0 < \Omega T/2 < \pi$, $\sin\Omega T/2 > 0$. Then from (2.194) we get

$$u^0(t) = -U_0\,\mathrm{sgn}\left[\sin\Omega(T/2 - t)\right], \tag{2.197}$$

and the oscillation period T is determined by the relation

$$|\Delta| = \frac{2U_0}{r}\frac{1 - \cos\Omega T/2}{\Omega^2\sin^2\Omega T/2}|\cos\Omega T/2|\,.$$

As a result of obvious transformations, we get the equation for determination of T

$$\cos\Omega T/2 = -\left[1 + 2U_0/r\Omega^2\Delta\right]^{-1}. \tag{2.198}$$

Obviously, the control (2.197) realizes in the system with a clearance ($\Delta > 0$) a regime close to a resonant one for any excitation intensity U_0; in systems with a press fit a quasi-resonant regime can be realized only for $2U_0 > r\Omega^2|\Delta|$, i.e., the control should be sufficient for overcoming a pressing force.

2.4.4
Control of Non-Autonomous Quasi-Resonant Systems

Suppose that the resonant regime in the system is supported by a parametric excitation, including also the control excitation

$$\left[D_0(p) + \varepsilon D_1(p)\right]x = \varepsilon g(t+\varphi, x, u), \tag{2.199}$$

a time-axis begin coincides with the impact moment, φ is an impact phase. Let us optimize the system with respect to the quality criterion (2.172).

An approximating scheme for a construction of the optimal control $u_*(t,\varepsilon)$ is analogous to the given above. We will search for successive approximations $\tilde{u}^l(t)$ minimizing at each stage the functional (2.152) on the trajectories of the system

$$\tilde{x}^l(t) = -J^l\chi_1^0(t) - \varepsilon\int_0^T\left\{\chi_1^0(t-s)g\left[s+\varphi, \tilde{x}^l(s), \tilde{u}^l(s)\right] + \chi_1^1(t-s)\tilde{x}^{l-1}(s)\right\}ds,$$

$$J^l = (1+R)\dot{\tilde{x}}_-^l(0), \quad \tilde{x}^l(0) = \Delta, \quad R = 1 - \varepsilon r. \tag{2.200}$$

It can be shown that in addition to introduced assumptions, it is necessary that the generating problem has the unique isolated solution φ^0.

Let us construct the solution of the generating problem: let us find the control $u^0(t)$ minimizing the functional (2.179) on the trajectory (2.177).

By the obvious transformations the isoperimeter condition (2.179) is reduced to the form

$$\int_0^T \chi_{1s}^0(T-s)g\left[s+\varphi, x^0(s), u(s)\right]ds = -\mu_1 J_0, \tag{2.201}$$

and the Lagrange function of the generating problem gets the form

$$L^0(u,\lambda) = -f_1^0(s,x^0) - f_2^0(u(s)) + \lambda\left\{\chi_{1s}^0(T-s)g\left[s+\varphi, x^0(s), u(s)\right] + \mu J_0 T^{-1}\right\}. \tag{2.202}$$

The optimal control is

$$u^0(s) = \arg\max L^0(u,\lambda)/u \in U\,; \tag{2.203}$$

the condition (2.201) serves for determination of the parameter λ.

Let, for instance, $\psi(u) = \frac{\alpha}{2}u^2$ and the set U be open. Then the control $u^0(s)$ is determined by the relation

$$u^0(s) = \alpha^{-1}\lambda\chi_{1s}^0(T-s)g_u\left[s+\varphi, x^0(s), u^0(s)\right]. \tag{2.204}$$

Let us study some particular cases.

1) The control of the parametrically excited system

$$g(t+\varphi, x, u) = -G(t+\varphi)x + u(t), \tag{2.205}$$

where $G(t)$ is a T-periodic function. Then the control u^0 is expressed by Eq. (2.190), and Eq. (2.201) gets the form

$$J_0(\mu_1 + \gamma(\varphi)) = \lambda\alpha^{-1}k_{12},$$

where coefficients μ_1 and k_{12} are expressed by Eqs. (2.182) and (2.191), respectively, and

$$\gamma(\varphi) = \int_0^T \chi_{1s}^0(T-s)\chi_1^0(s)G(s+\varphi)ds\,. \tag{2.206}$$

Thus,

$$u^0(s) = -J_0(\mu_1 + \gamma(\varphi))k_{12}^{-1}\chi_{1s}^0(T-s)\,. \tag{2.207}$$

Eq. (1.77) which determines the phase with account for (2.192), (2.194) can be reduced to the form

$$\int_0^T \chi_{1s}^0(T-s)\chi_1^0(s)G_\varphi(s+\varphi)ds = 0\,,$$

or, according to (2.196),

$$\gamma_\varphi(\varphi) = 0\,. \tag{2.208}$$

In particular, if $G(t) = a_1\cos\omega t$, then $\gamma(\varphi) = a_1\sin\omega t$, and Eq. (2.208) gets the simple form: $\cos\omega\varphi = 0$.

2) The parametrically controlled system

$$g(t,x) = -u(t)x\,. \tag{2.209}$$

Then still $x_0(t) = -J_0\chi_1^0(t)$, i.e.,

$$u^0(s) = \lambda\alpha^{-1}\chi_{1s}^0(T-s)x^0(s) = -\lambda\alpha^{-1}\chi_1^0(s)\chi_{1s}^0(T-s)J_0\,. \tag{2.210}$$

Excluding λ with the help of (2.201), we get

$$u^0(s) = -\mu_1 \chi^0_{1s}(T-s)\chi^0_1(s)\kappa\,,$$

where

$$\kappa = \int_0^T \left[\chi^0_1(s)\chi^0_{1s}(T-s)\right]^2 ds\,.$$

Let us separately consider a situation of a control supporting an optimal resonant regime with a constraint $|u| \le U_0$. A control in a system with the excitation (2.205) keeps the form (2.193). Substituting (2.193) into (2.201), we get

$$-J_0\left(\mu_1 + \gamma(\varphi)\right) = -U_0 \int_0^T \left|\chi^0_{1s}(T-s)\right| ds \operatorname{sgn}\lambda\,. \tag{2.211}$$

Thus, if $\mu_1 + \gamma(\varphi) > 0$ then $\operatorname{sgn}\lambda > 0$ and

$$u^0(s) = -U_0 \operatorname{sgn}\chi^0_{1s}(T-s)\,. \tag{2.212}$$

If $\mu_1 + \gamma(\varphi) < 0$ then $\operatorname{sgn}\lambda < 0$,

$$u^0(s) = U_0 \operatorname{sgn}\chi^0_{1s}(T-s)\,. \tag{2.213}$$

In the parametrically controlled system we will get, owing to (2.202), (2.203), (2.209),

$$u^0(s) = -U_0 \operatorname{sgn}\left[\lambda\chi^0_{1s}(T-s)\chi^0_1(s)\right]. \tag{2.214}$$

Substituting (2.214) into (2.211), we get

$$\mu_1 J_0 = -U_0 \int_0^T \left|\chi^0_{1s}(T-s)\right| \operatorname{sgn}\chi^0_1(s)\,ds \operatorname{sgn}\lambda\,. \tag{2.215}$$

Eq. (2.215) serves for determination of the period T and $\operatorname{sgn}\lambda$.

2.4.5 Approximate Optimal Control Synthesis for Vibroimpact Systems

The above obtained solution for the optimal control problem makes it possible, to synthesize a self-oscillatory system in which an optimal movement law for a striking element is realized. This is achieved via exclusion of the time parameter from the expression for $u^0(s)$.

It can be easily shown that in problems for systems with an autonomous control the optimal control $u^0(s)$ can be obtained as a function of $\chi^0_1(s)$, $\chi^0_{1s}(s)$: $u^0 =$

$u^0\left[\chi^0(s), \chi^0_{1s}(s)\right]$. Replacing the arguments

$$\chi^0_1(s) = J_0^{-1} x^0(s), \quad \chi^0_{1s}(s) = J_0^{-1} \dot{x}^0(s), \tag{2.216}$$

we get $u^0(s)$ in the form of the synthesis depending of the coordinates and velocities in a generating approximation

$$u = u^x\left[x^0(s), \dot{x}^0(s)\right]. \tag{2.217}$$

Finally, if $x^0(s)$, $\dot{x}^0(s)$ are replaced with $x(s)$, $\dot{x}(s)$ then the control

$$u = u^x\left[x(s), \dot{x}(s)\right] \tag{2.218}$$

will be approximately optimal in the following sense [134]: If it is substituted into the initial system (2.199), then the corresponding trajectory is situated in the ε – zone of the optimal trajectory, and the value of the functional differs from the optimal by $O(\varepsilon)$.

In the self-oscillatory system, the period T is not fixed but is determined by structural features of the system, so Eq. (2.194) should be added to the equations of the maximum principle.

Let us consider some examples (comp. [14, 44]).

1) The synthesis of the self-oscillatory system. Let $g(s, x, u) = -u(s)$. We will find the control $u(s)$: $|u(s)| \le U_0$, forming the self-oscillatory regime with the maximal impulse. From (2.212) we have

$$u^0(s) = -U_0 \operatorname{sgn} \chi^0_{1s}(T - s) = U_0 \operatorname{sgn} \dot{x}^0(s),$$

i.e.,

$$u^x(s) = U_0 \operatorname{sgn} \dot{x}(s). \tag{2.219}$$

2) The synthesis of the autoparametric system. Let $g(t, x, u) = -ux$, $|u| \le U_0$. Then from (2.214) we get

$$u^0(s) = -U_0 \operatorname{sgn} \lambda \operatorname{sgn}\left[\chi^0_{1s}(T - s)\right] \operatorname{sgn}\left[\chi^0_1(s)\right],$$

i.e.,

$$u(s) = -U_0 \operatorname{sgn} \lambda \operatorname{sgn}\left[x(s) \dot{x}(s)\right], \tag{2.220}$$

and $\operatorname{sgn} \lambda$ is determined from Eq. (2.216). As far as $\operatorname{sgn} \Delta = -\operatorname{sgn} \chi^0_1(0)$, then, accounting for (2.216), we will get

$$u^x(s) = U_0 \operatorname{sgn} \lambda \operatorname{sgn}\left[x(s) \dot{x}(s)\right]. \tag{2.221}$$

2.4.6
Control of Asymmetric Vibroimpact Systems

Let us indicate the distinctive features of control calculations for a vibroimpact system with asymmetrically situated limiters. For the sake of briefness, let us consider a system linear between the impacts, dynamics of which is described by Eq. (2.164) (for $g = 0$) and by the impact conditions (2.35)

$$\begin{aligned} &t = 0, \quad x = \Delta_1, \quad \dot{x}_+ = -(1 - \varepsilon r_1)\dot{x}_-, \\ &t = \varphi, \quad x = -\Delta_2, \quad \dot{x}_+ = -(1 - \varepsilon r_2)\dot{x}_- \end{aligned} \tag{2.222}$$

(the time axis begin coincides with the impact against the right limiter).

The integral equation of the periodic regime has the form [comp.(2.38)]

$$x(t) = -J_1\chi_1^0(t) + J_2\chi_1^0(t-\varphi) + \varepsilon\int_0^T \left[\chi_1^u(t-s)u(s) - \chi_1^1(t-s)u(s)\right]ds, \tag{2.223}$$

where the periodic Green's functions χ_1^0, χ_1^1 and χ_1^u are determined by relations (2.167).

The discussed scheme of successive approximations is used for a search of the optimal control. The generating solution is determined by Eq. (2.61)

$$\begin{aligned} &x^0(t) = -J_1^0\chi_1^0(t) + J_2^0\chi_1^0(1-\varphi^0), \qquad \varphi^0 = T/2, \\ &-J_1^0\chi_1^0(0) + J_2^0\chi_1^0(T/2) = \Delta_1, \\ &-J_1^0\chi_1^0(T/2) + J_2^0\chi_1^0(0) = -\Delta_2. \end{aligned} \tag{2.224}$$

The impulses J_1^0, J_2^0 are expresses by Eqs. (2.160), (2.161).

It was shown in Section 2.2 that a double-impact regime in the asymmetric system is degenerated into the one-impact regime in a case of the insufficient initial energy supply. In such a case, a solution describing a one-impact regime with impact against respective limiter should be chosen as a generating solution.

Let us construct an equation of the first approximation according to the scheme of Theorem 2.1. Let us write

$$x^1(t) = -J_1^1\chi_1^0(t) + J_2^1\chi_1^0(t-\varphi^1) + \varepsilon\theta_u(t) + \varepsilon\theta_1^0(t), \tag{2.225}$$

where

$$\begin{aligned} &\theta_u(t) = \int_0^T \chi_1^u(t-s)u(s)ds, \\ &\theta_1^0(t) = -\int_0^T \chi_1^1(t-s)x^0(s)ds. \end{aligned} \tag{2.226}$$

Substituting (2.225) into (2.222), we get equations, determining the first approximation of the impulses and the impact phase

$$-J_1^1\chi_1^0(0)+J_2^1\chi_1^0(T/2)=\Delta_1-\varepsilon\theta_u(0)-\varepsilon\theta_1^0(0),$$
$$-J_1^1\chi_1^0(T/2)+J_2^1\chi_1^0(0)=-\Delta_2-\varepsilon\theta_u(T/2)-\varepsilon\theta_1^0(T/2), \tag{2.227}$$

and

$$(\varepsilon r_1/4)J_1^0+\dot\chi_1^0(\varphi^1)J_2^0=\varepsilon\dot\theta_u(0)+\varepsilon\dot\theta_1^0(0),$$
$$\dot\chi_1^0(\varphi^1)J_1^0-(\varepsilon r_2/4)J_2^0=\varepsilon\dot\theta_u(T/2)+\varepsilon\dot\theta_1^0(T/2). \tag{2.228}$$

In (2.227), (2.228) it is accounted that $J_{1,2}^1=J_{1,2}^0+O(\varepsilon)$, $\varphi^1=T/2+O(\varepsilon)$. Then $\chi_1^0(\varphi^1)=\chi_1^0(T/2)+O(\varepsilon^2)$, but $\dot\chi_1^0(\varphi^1)=O(\varepsilon)$.

Let us calculate the derivatives $\dot\theta_1^0$. From (2.224), (2.226) we have

$$\dot\theta_1^0(0)=-\gamma_1J_1^0+\gamma_2J_2^0,\qquad \dot\theta_1^0(T72)=-\gamma_2J_1^0+\gamma_1J_2^0, \tag{2.229}$$

where

$$\gamma_1=\int_0^T\chi_{1s}^1(T-s)\chi_1^0(s)ds,$$
$$\gamma_1=\int_0^T\chi_{1s}^1(T/2-s)\chi_1^0(s)ds \tag{2.230}$$

[coefficient γ_1 coincides with the one calculated in (2.180)]. Substituting (2.229) into (2.228), we get

$$\varepsilon(r_1/4+\gamma_1)J_1^0+\left(\dot\chi_1^0(\varphi^1)-\varepsilon\gamma_2\right)J_2^0=\varepsilon\dot\theta_u(0),$$
$$\left(\dot\chi_1^0(\varphi^1)+\varepsilon\gamma_2\right)J_1^0-\varepsilon(r_2/4+\gamma_1)J_2^0=\varepsilon\dot\theta_u(T/2). \tag{2.231}$$

Excluding $\dot\chi_1^0(\varphi^1)$ from (2.231) and taking into account (2.226), we will get

$$\int_0^T K^u(T,s)u(s)ds=\mu(T), \tag{2.232}$$

where

$$K^u(T,s)=J_1^0\chi_{1s}^u(T-s)-J_2^0\chi_{1s}^u(T/2-s),$$
$$\mu(T)=-\left[(r_1/4+\gamma_1)\left(J_1^0\right)^2+(r_2/4+\gamma_1)\left(J_2^0\right)^2-2\gamma_2J_1^0J_2^0\right]. \tag{2.233}$$

It is easy to show that Eqs. (2.232), (2.233) are reduced to the form of (2.181), (2.182) for $J_2^0 = 0$ and to respective expressions for a system with symmetric limiters for a case of $J_1^0 = J_2^0$. In the latter case it is accounted that

$$\gamma = \gamma_1 - \gamma_2 = \int_0^{T/2} \chi_{2s}^1 (T/2 - s) \chi_2^0 (s) ds$$

is a reduced work of dissipative forces in the system with asymmetric limiters during the half-period.

Thus, the movement of the asymmetric system in the first approximation is described by (2.232). The optimal control construction is implemented in the same way as in Section 2.4.1, with the kernel $\chi_{1s}^u (T - s)$ replaced with $K^u (T,s)$.

Other variants of asymmetric systems, e.g., changing their characteristics at impact (by change of the mass of a striking element) are studied similarly.

3 The Averaging Method in Oscillation Control Problems

Main methods of construction of optimal periodic regimes are given in the previous Sections. If a system is either linear or quasi-linear, then the method of integral equations allows us to overcome the hardships linked with the oscillatory character of the solution and a high order of the system. However, these advantages are lost even in a study of resonant regimes, since a generating equation becomes considerably non-linear (Section 1.6). At the same time, just the resonant regimes are especially important for applications since they allow the realization of optimal stationary processes with a sufficient high-speed action and small energy supply.

Asymptotic methods combining both traditional approaches of the optimal control theory and well-posed approximate methods of the oscillation theory are more suitable for an analysis of controlled regimes supported by low-level guiding excitations [6, 40, 41, 103, 110, 111, 134, 135, 155, 172].

An application of an averaging procedure to weak-controlled systems is given in detail in monographs [6, 134] and numerous papers (voluminous references are given in [135]). The effect of weak excitations and perturbing factors was studied for a case of a large time interval, where considerable changes of phase coordinates under the weak action took place.

Such a formalization is justified, for instance, for motion control problems in a weak force field. The presence of oscillatory elements in such a system allows the reduction of motion equations to a standard form and the use of the averaging procedure. If an aim of the problem is not a free-oscillation control but an 'imposition' of a given motion regime on the system, then considerable changes of phase coordinates need considerable time.

A situation in quasi-resonant systems is principally different. A system motion is considered to be close to free oscillations, and a control only compensates an effect of dissipative and perturbing factors. And considerable changes in such a case are achieved during the time intervals of the order of the free-oscillation period.

In Section 3.1, control problems on finite time intervals are examined. The difference between weak-controlled systems and systems of a quasi-resonant type are discussed. A solution scheme for optimization problems for weak-controlled systems, linked with an averaging of equations of the maximum principle, is based on results of [4-6, 9, 134], so the main ideas in Section 3.1 are only formulated and discussed. Some illustrative examples are studied: a problem of an optimal high-speed action in a weak-controlled system and the problem of an optimal displacement in a system of quasi-resonant type.

In Section 3.2 the averaging method in periodic control problems is given [76].

A detailed bibliography dedicated to the periodic control can be found in reviews [123, 124, 157, 163, 184, 185, 187]. A search for an optimal control is based either on a solution of a two-point problem of the maximum principle [187] or on a solution of a dynamic programming equation [158, 183]. Such problems have rarely an analytical solution; so the approximate methods for determination of a quasi-optimal control are of special interest.

A solution for a linear system with a mean–square quality criterion was constructed in the form of a linear feedback, coefficients of which satisfied a periodic Riccati equation [151, 183] with a subsequent averaging of the Riccati equation.

The averaging method was chosen in [111] as a basis for a solution of periodic control problems; the averaging was carried out directly in motion equations, and the control was submitted to relations of the method of moments.

Another approach to the oscillation control, naturally following from results of Section 3.1, is given below. The motion equations are given in the standard form, and the system of equations of the maximum principle also having the standard form is constructed. A stationary solution of the averaged system corresponds to a periodic control. Thus, a procedure is reduced not to a solution of the two-point problem, but to a search for roots of a system of algebraic equations.

In Section 3.3 analogous problems for vibroimpact systems are examined.

The main theorems of the averaging methods are considered to be known and are given in Appendix (Section A.3).

3.1 Optimal Control for Finite Time Interval. Problems of the Optimal High–Speed Action

3.1.1 Motion Equations for Systems with Weak Control

We will demonstrate the specific features of control problems for oscillatory systems with an example of the simplest model of a linear system.

Let dynamics of the system be described by the equation

$$\ddot{z} + Az + 2\varepsilon B\dot{z} = \varepsilon K u \,, \tag{3.1}$$

where $z \in R_n$ is a vector of generalized coordinates, $u \in R_m$ is a control vector, A is a positive-determined matrix, all the eigenvalues p_j of which are on the imaginary axis, $p_j^2 = -\Omega_j^2$, Ω_j are eigenfrequencies of the system, $j = 1, \ldots, n$; B is a positive-determined matrix of dissipation coefficients, K is a matrix of amplifying coefficients with dimension $n \times m$, ε is a small parameter. It is supposed that $\Omega_j \neq \Omega_l$, $j \neq l$.

Suppose that the aim of the problem is a transition of the system (3.1) from an initial state

$$z_j(t_0)=\xi_j, \quad \dot{z}_j(t_0)=\eta_j, \quad j=1, ..., n, \tag{3.2}$$

to a final one

$$z_r(t_f)=\xi_n, \quad \dot{z}_l(t_0)=v_l, \tag{3.3}$$
$$r=1, ..., m_1 \le n; \quad l=1, ..., m_2 \le n,$$

in either minimal or fixed time $(t_f - t_0)$.

Let us note that in real problems the final states can be fixed with respect not to all but only to some generalized coordinates, which fix, for instance, a position and velocity of an actuator.

Obviously, the generating system $\ddot{z}^0 + Az^0 = 0$ for $\varepsilon = 0$ has a solution of the form $z_j^0(t)=\sum_{k=1}^{n} A_{jk} \cos\Omega_k(t+\varphi_k)$, $j=1, ..., n$, describing free oscillations under initial conditions (3.2). Depending on the character of boundary conditions (3.3), two types of motion can be realized in a weak-controlled system.

If there exists such a point $t=t_*$ that for $t_f = t_* + O(\varepsilon)$ boundary conditions (3.3) are within the $\varepsilon-$ zone of a set $\{z_r^0(t_*), \dot{z}_r^0(t_*)\}$, then the motion of the system (3.1) is treated as free oscillations supported by an external force. The aim of a weak control is then a small motion correction in the $\varepsilon-$ zone of the eigentrajectory of the generating non-controlled system. In this case a displacement over a finite distance is realized with the help of the weak control in a finite time. If a problem of the optimal high-speed action is being solved then the non-fixed moment t_f of the end of the process is submitted to the condition $|t_f - t_*| = O(\varepsilon)$.

If the boundary conditions (1.3) are not in the $\varepsilon-$ zone of the eigentrajectory of the generating system for any finite values t_f, then the searched displacement under weak forces can be realized only over an asymptotically large interval of time, $O(\varepsilon^{-1})$.

The first case is characteristic for so-called resonant systems [17, 18], the second case usually describes a motion of bodies in weak force fields, and was studied in detail in [6, 134].

Below an application of the averaging procedure to weak-controlled systems is given. An entire discussion follows [6, 134], but results are given in the form more suitable for analysis of oscillatory systems, studied in Sections 3.2, 3.3.

3.1.2
Problem Formulation. General Equations

Let us examine a more general problem of motion of a weak-controlled oscillatory system.

Let equations of the system motion be reduced to a standard form with one rotatory phase:

$$\dot{y} = \varepsilon f_1(\psi, y, u), \quad \dot{\psi} = \omega(y) + \varepsilon f_2(\psi, y, u), \tag{3.4}$$

$$y(t_0) = a_0, \quad \psi(t_0) = \psi_0. \tag{3.5}$$

Here $y \in R_n$ is a vector of phase variables, ψ is a scalar fast phase, $u \in R_m$ is a vector of controlling excitations, $\omega(y)$ is an angular frequency, $\omega(y) \ge \omega_0 > \varepsilon$, ε is a small parameter. We will search for a solution from the class of measurable controls, satisfying the constraint $u \in U$. Here and below U is a bounded domain in R_m. It is assumed, that the solution of the problem (3.4), (3.5) exists and is unique for any admissible control. It is also assumed that functions f_1, f_2 are piecewise continuous and 2π – periodic in ψ uniformly with respect to y, u and are sufficiently smooth with respect to y, u for $y \in Y$, $u \in U$, $-\infty < \psi < \infty$. Smoothness conditions and other constraints to system variables are specified in the course of solution.

Let us give a main scheme of construction of the approximated solution. We limit our considerations to one optimal control problem. Let us consider that the system should be transited from the initial state (3.5) to a final one, which is determined by the relation

$$G(y(t_f)) = 0, \quad G = (G_1, \ldots, G_l), \quad 1 \le l \le n, \tag{3.6}$$

in time $t_f - t_0$ in such a way that the functional

$$\Phi_\varepsilon = g(y(t_f)) / u \in U \tag{3.7}$$

becomes its maximum value. Functions G and g are supposed to be two-times differentiable with respect to y and such, that for all $0 < \varepsilon \le \varepsilon_0$ a solution of the problem (3.4) – (3.7) exists and is unique.

The minimization problem for the functional

$$\Phi(u) = \int_{t_0}^{t_f} f_0(\psi, y, u) dt \tag{3.8}$$

(Bolza problem) can be reduced to (3.7), if the functional

$$\Phi_\varepsilon(u) = \varepsilon \int_{t_0}^{t_f} f_0(\psi, y, u) dt \tag{3.9}$$

is considered instead of (3.8), and an additional phase variable y_{n+1} determined by relations

$$\dot{y}_{n+1} = \varepsilon f_0(\psi, y, u), \quad y_{n+1}(t_0) = 0, \tag{3.10}$$

is introduced. Increasing a dimension of the phase vector by one, we reduce the minimization problem for an integral functional to the Mayer problem (3.7).

Let us note that the system (3.4) and functionals (3.7), (3.9) do not depend on t. So, the problem can be simplified if the phase ψ can be chosen as an independent variable. Introducing a new 'slow' variable $\varphi = \varepsilon\psi$ and using only $O(\varepsilon)-$ terms, we get

$$\frac{dy}{d\varphi} = \omega^{-1}(y) f_1(\varphi/\varepsilon, y, u), \quad y(\varphi_0) = a_0. \tag{3.11}$$

Boundary conditions (3.6) and the functional (3.4) can be written in the form

$$G\big(y(\varphi_f)\big) = 0, \tag{3.12}$$

$$\Phi_\varepsilon(u) = g\big(y(\varphi_f)\big). \tag{3.13}$$

The system (3.4) is autonomous, and the phase ψ is determined with the accuracy up to an additive constant. Hence, we can consider $\psi_0 = \varphi_0/\varepsilon = 0$ in order to shorten the form of boundary conditions. A final phase $\psi_f = \varphi_f/\varepsilon$ is linked with $t_f = t_f/\varepsilon$ by the relation

$$\tau_f - \tau_0 = \int_0^{\varphi_f} \omega^{-1}(y) d\varphi, \tag{3.14}$$

i.e., the problem (3.11) – (3.14) can be treated as the problem with a free right-hand end. In this case it is assumed that for any admissible control u and $0 < \varepsilon \le \varepsilon_0$ a reduction of the initial system (3.4) to equations (3.11), (3.14) holds with an error $O(\varepsilon^2)$, and the solution of the optimal control problem for the system (3.11) – (3.14) exists, is unique and determined by equations of the maximum principle.

It follows form Section A.2 that a control $u_*(\psi, y)$ realizing the minimum of the functional (3.13) on the trajectory of the system (3.11) is quasi-optimal with respect to the initial system (3.4); so, the initial problem can be replaced in the first approximation by the optimization problem for the system (3.11).

Let us use the averaging method for a solution of the problem (3.11) – (3.14) and give necessary accuracy estimates for the solution [6, 134].

At first, let us consider a problem with a fixed end moment t_f. Then a Hamilton's function of the problem under study has a form (see Section A.1)

$$H_1 = \omega^{-1}(y)\big[H(\varphi/\varepsilon, y, u, p) + h\big], \tag{3.15}$$

$$H = \big(p, f(\varphi, \varepsilon, y, u)\big),$$

where $h = \text{const}$, a Lagrange multiplier p satisfies the equation

$$\frac{dp}{d\varphi} = -\frac{\partial H_1}{\partial y} \tag{3.16}$$

and a transversal condition

$$p(\varphi_f) = [\rho' \partial G/\partial y - \partial g/\partial y]_{\varphi=\varphi_f}, \tag{3.17}$$

where $\rho = (\rho_1, ..., \rho_l)$ is a constant vector of Lagrange multipliers. The condition (3.12) and Eq. (A.20)

$$H_1|_{\varphi=\varphi_f} = 0 \tag{3.18}$$

serve additionally for determination of the constant h and phase φ_f.

Let us show that the condition (3.17) can be treated as a link between boundary conditions $p(\varphi_f)$ and $y(\varphi_f)$. Write (3.17) in the form

$$p_r(\varphi_f) = \left[\sum_{k=1}^{l} \rho_k \frac{\partial G_k}{\partial y_r} - \frac{\partial g}{\partial y_r}\right]_{\varphi=\varphi_f}, \qquad r = 1, ..., n.$$

Suppose that the matrix $[\partial G/\partial y]_{\varphi=\varphi_f}$ has a maximal rang l and present the vector ρ as the solution of the linear system of the first l equations

$$\rho = \left\{\left[\tilde{p} + \frac{\partial g}{\partial \tilde{y}}\right]\left(\frac{\partial G}{\partial \tilde{y}}\right)^{-1}\right\}_{\varphi=\varphi_f}.$$

Here $\tilde{p}$ and $\tilde{y}$ are l-dimensional vectors including the first l components of p and y, respectively. Then

$$p_s(\varphi_f) = \sum_{k=1}^{l} \rho_k(\tilde{p}, \tilde{y}) \frac{\partial G_k}{\partial y_s} - \frac{\partial g}{\partial y_s}, \qquad s = l+1, ..., n, \tag{3.19}$$

i.e., we have $n-l$ relations linking $p(\varphi_f)$ and $y(\varphi_f)$, and l conditions (3.12) including only $y(\varphi_f)$. Thus, we got a two-point boundary-value problem with additional conditions (3.12), (3.18).

Suppose that the optimal control u_* is uniquely determined from the maximum principle

$$u_* = \arg\max_{u \in U} H_1(\psi, y, u, p, h) = \arg\max_{u \in U} H_1(\psi, y, u, p), \tag{3.20}$$

$$u_* = U(\psi, y, p).$$

Let also functions

$$f_1(\psi, y, U(\psi, y, p)) = f_1^u(\psi, y, p),$$
$$H^u(\psi, y, p) = (p, f_1^u(\psi, y, p)), \tag{3.21}$$
$$H_1^u(\psi, y, p, h) = \omega^{-1}(y)[H^u(\psi, y, p) + h]$$

be such that right-hand parts of Eqs. (3.11), (3.16) and boundary conditions (3.19) satisfy conditions of Theorem A.5. Then the solution of the problem (3.11), (3.17) are approximated by the solution of a shortened system

$$\frac{dy^0}{d\varphi} = \omega^{-1}(y^0) f^0(u^0, p^0), \quad \frac{dp^0}{d\varphi} = -\frac{\partial}{\partial y^0} H_1^0(y^0, p^0, h^0) \tag{3.22}$$

with boundary conditions

$$p^0(\varphi_f) = \left[(\rho^0, \partial G/\partial y^0) - \partial g/\partial y^0\right]_{\varphi=\varphi_f}, \quad G(y^0(\varphi_f)) = 0 \tag{3.23}$$

$$H_1^0(\varphi_f) = 0, \tag{3.24}$$

where

$$f^0(y, p) = \frac{1}{2\pi}\int_0^{2\pi} f_1^u(\psi, y, p) d\psi,$$
$$H^0(y, p) = \frac{1}{2\pi}\int_0^{2\pi} H^u(\psi, y, p) d\psi, \tag{3.25}$$
$$H_1^0(y, p, h) = \omega^{-1}(y)[H^0(y, p) + h],$$

and for $\varepsilon \in (0, \varepsilon]$, $\varphi \in [0, L]$ the estimates

$$|y_*(\varphi, \varepsilon) - y^0(\varphi)| \le C\varepsilon, \quad |p(\varphi, \varepsilon) - p^0(\varphi)| \le C\varepsilon, \tag{3.26}$$
$$|h - h^0| \le C\varepsilon$$

hold true. Here and below C, C_j are constants which do not depends on ε. For a proof of the latter estimate Eq. (3.14) should be written in the form

$$d\tau/d\varphi = \omega^{-1}(y), \quad \tau(\varphi_f) = \tau_f - \tau_0$$

and h be treated as a Lagrange multiplier corresponding to the variable τ in the expanded system of variables $z = (y, \tau)$, $q = (p, h)$. In such a case we have for h the equation

$$dh/d\varphi = 0.$$

Let us explain the condition (3.24). From the suggestion about existence of the solution for the optimal control problem (3.11) – (3.14) it follows that Eq. (3.18) has at least one root $\varphi_f = \varphi$. At the same time it follows from the form of Eq. (3.18) that the root is determined non-uniquely since H^1 as the function of φ_f oscillates with a frequency $\propto \varepsilon^{-1}$ and an amplitude of an order of unity. It is shown in [6, 134] that equations of the form (3.18) have a number of roots of the order of ε^{-1} on a final interval $[\varphi_1, \varphi_2]$, and a distance between neighboring roots is of the order of ε. For a solution of the optimal control problem, the minimum of the functional (3.13) should be found on the set $\{\varphi_*\}$ of solutions of Eq. (3.18).

Obviously, an accuracy of the solution is not changed, if the root φ_* is determined with an error $O(\varepsilon)$. Consider an additional minimization problem (with respect to φ_f) for the functional $g(y^0(\varphi_f))$ on the trajectory of the system (3.22) with boundary conditions (3.23) for construction of an approximated solution. Necessary conditions for minimum of $g(y^0(\varphi_f))$ constructed with account for (3.22), (3.23) lead to Eq. (3.24).

Let Eq. (3.24) have a unique root φ_*^0. In analogy to Section A.2, it is easy to obtain the necessary estimate

$$0 \le g(y(\varphi_*^0)) - g(y(\varphi_*)) \le C\varepsilon .$$

Thus, the boundary condition (3.18) can be replaced by the averaged condition (3.24) with the adopted accuracy.

A detailed study of the roots of Eq. (3.18) and of conditions of the global minimum of the functional (3.13) is given in [6, 134].

Let us show that the control

$$u_0(\psi, \varepsilon) = U(\psi, y^0(\varphi), p^0(\varphi)), \quad \varphi = \psi\varepsilon \tag{3.27}$$

is quasi-optimal with respect to the system (3.11), i.e., for $0 < \varepsilon \le \varepsilon_1$

$$|\tilde{y}(\varphi, \varepsilon) - y_*(\varphi, \varepsilon)| \le C_1 \varepsilon , \tag{3.28}$$

where $\tilde{y}(\varphi, \varepsilon)$ is a solution of Eq. (3.11) for $u = u_0(\psi, \varepsilon)$. Write for a proof an obvious relation

$$|\tilde{y}(\varphi, \varepsilon) - y_*(\varphi, \varepsilon)| \le |\tilde{y}(\varphi, \varepsilon) - y^0(\varphi)| + |y_*(\varphi, \varepsilon) - y^0(\varphi)|, \tag{3.29}$$

where $y^0(\varphi)$ is a solution of the shortened system (3.20).

The second term in (3.29) is submitted to the estimate (3.26). Besides, it is obvious that submitting a control in the form (3.27) into the right-hand part of Eq. (3.11) and averaging, we get the analogous estimate for the function $\tilde{y}$

$$|\tilde{y}(\varphi, \varepsilon) - y^0(\varphi)| \le C_2 \varepsilon$$

for $0 < \varepsilon \le \varepsilon_2$. Thus, for $0 < \varepsilon \le \varepsilon_1 = \min(\varepsilon_1, \varepsilon_2)$

$$\left|\tilde{y}(\varphi,\varepsilon) - y_*(\varphi,\varepsilon)\right| \le C_1\varepsilon, \qquad C_1 \le C + C_2 .$$

If the system (3.4) can be reduced to the standard form

$$\dot{y} = \varepsilon f(t, y, u), \qquad \dot{t} = 1, \tag{3.30}$$

and the function f is either periodic or continuously quasi-periodic with respect to t, then all transformations remain with replacement of ψ with t. The shortened system (3.22) gets the form

$$\frac{dy^0}{d\tau} = f^0(y^0, p^0), \qquad \frac{dp^0}{d\tau} = \left(p^0, f_y^0(y^0, p^0)\right) \tag{3.31}$$

where

$$f^0(y,p) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, y, U(t, y, p))dt .$$

The boundary conditions (3.23) remain with replacement of φ with $\tau = \varepsilon t$.

Now we consider a problem of an optimal high-speed action. Neglecting terms of the higher order of smallness, let us re-write (3.4), (3.7) in the form

$$\frac{dy}{d\varphi} = \omega^{-1}(y) f_1(\varphi/\varepsilon, y, u), \qquad \frac{d\tau}{d\varphi} = \omega^{-1}(y), \tag{3.32}$$

$$g_1\left(z(\varphi_f)\right) = g\left(y(\varphi_f)\right) + \tau(\psi_f) = \min, \tag{3.33}$$

where $z = (y, \tau)$ is an expanded vector of variables. In contrast to (3.13), the problem's functional directly depends on τ. The condition (3.12) remains without changes.

The Hamilton's function of the problem under study keeps its form (3.15). The multiplier p still satisfies Eq. (3.16) with the boundary condition (3.17) or (3.19); the equation

$$\dot{h} = -\partial H/\partial \tau = 0 \tag{3.34}$$

and the boundary condition

$$h(\varphi_f) = \left[(\rho, \partial G/\partial \tau) - \partial g_1/\partial \tau\right]_{\varphi=\varphi_f}, \tag{3.35}$$

serve for determination of the multiplier h, i.e., $h = -1$, and the Hamilton's function H_1 is transformed to the form

$$H_1 = \omega^{-1}(y)\left[H(\varphi/\varepsilon, y, p, u) - 1\right]. \tag{3.36}$$

For determination of the moment φ_f of the process end, a transversal condition for the right-hand end is supplemented by the equality

$$H_1(\varphi/\varepsilon, y, p, u_*)\Big|_{\varphi=\varphi_f} = 0, \tag{3.37}$$

or, owing to (3.36),

$$H(\varphi/\varepsilon, y, p, u_*)\Big|_{\varphi=\varphi_f} = 1. \tag{3.38}$$

A solution of the problem (3.12), (3.32), (3.36) can be still approximated by the solution of the shortened system (3.38) with the boundary conditions (3.23). The condition (3.24) is reduced to the form

$$H^0(y^0, p^0)\Big|_{\varphi=\varphi_f} = 1. \tag{3.39}$$

In some cases it is expedient to obtain a solution, depending directly not on the 'slow' phase φ but on the 'slow' time τ. In such a case a reverse replacement of variables can be carried out with the reduction of the shortened system to the form (3.31). The moment of the process end with the adopted accuracy is determined by the equation

$$H^0(y^0(\tau), p^0(\tau))\Big|_{\tau=\tau_*^0} = 1 \tag{3.40}$$

and is connected with the moment τ_* of the optimal high-speed action by the estimate $\tau_*^0 = \tau_* + O(\varepsilon)$.

The function (3.20) can be treated as a control synthesis law. Expressing the dependence $p^0(y^0(\varphi)) = P^0(\varphi, y^0)$ from Eq. (3.22) and making in (3.27) the substitution

$$\tilde{u}_0(\psi, y, \varepsilon) = U(\psi, y, P^0(\varphi, y)), \tag{3.41}$$

we get the control in the form of synthesis. As above, it can be shown that the control (3.41) is quasi-optimal with respect to the initial system with estimates $O(\varepsilon)$.

All the obtained estimates of the ε-optimality concerned also the reduced system (3.4). It is obvious that the controls (3.27) and (3.41) are also ε-optimal for the system (3.41).

The obtained results can be formulated in the following way.

Suppose that for any admissible control on the domains

$$y \in Y, \quad p \in P, \quad u \in U, \quad -\infty < \psi < \infty, \quad 0 < \varepsilon \le \varepsilon_0,$$

where Y, P are open domains in R_n, a set U does not depend on t, y, ψ, the following conditions hold:

1) functions $f_1(\psi, y, u(\psi, y))$, $f_2(\psi, y, u(\psi, y))$ are 2π-periodic and measurable with respect to ψ continuously with respect to y, continuously bounded for all ψ, y;

2) the functions f_1, f_2 are two-times continuously differentiable with respect to p, y and differentiable with respect to u; functions $\omega(y) \geq \omega_0 > 0$ are two-times continuously differentiable with respect to y;

Let further the optimal control $u_* = U(\psi, y, p)$ be uniquely determined from equations of the maximum principle, and for $u = U(\psi, y, p)$

3) the right-hand parts of Eqs. (3.11), (3.16) satisfy the conditions of Theorem A.5;

4) the boundary conditions (3.12), (3.17) and (3.19), determined by the functions $g(y)$, $G(y)$, satisfy the conditions of Theorem A.5;

5) there exists a unique solution $y^0 \in Y$, $p^0 \in P$, $\varphi_*^0 \in [0, L]$ of the averaged system of equations (3.22) – (3.24) [or (3.39)].

The boundedness of the function f_2 allows a replacement of the initial system (3.4) with the equivalent system (3.11), (3.14); smoothness conditions for the functions f_1, f_2 allow the use of the maximum principle; and together with conditions 3) – 5) they guarantee a closeness of solutions of the perturbed and averaged boundary–value problems.

Thus, the following Theorem holds:

Theorem 3.1. *For satisfied conditions* 1) – 5)

1. *A solution of the boundary–value problem of the maximum principle* (3.11) – (3.18) $[y_*(\varphi, \varepsilon), p(\varphi, \varepsilon), \varphi_*]$ *is in the* ε *-zone of the solution* $[y^0(\varphi), p^0(\varphi), \varphi_*^0]$ *of the averaged system* (3.22) – (3.24).

2. *For* $t \in [0, T/\varepsilon]$ *the estimates*

$$|y_*(t, \varepsilon) - y^0(\varphi)| \leq C\varepsilon, \quad |\tau_* - \tau_*^0| \leq C\varepsilon$$

hold. Here y_* *is a solution of the system* (3.4) *for* $u = u_*$, u_* *is an optimal control,* τ_* *is an optimal (or fixed) moment of the process end,* $\tau_*^0 = \tau(\varphi_*^0)$ *is a solution of* Eq. (3.14) *for* $y = y^0(\varphi)$, $\varphi_f = \varphi_*^0$.

3. *A control*

$$u_0(\psi, \varepsilon) = U(\psi, y^0(\varphi), p^0(\varphi)), \quad \varphi = \varepsilon\psi,$$

and a control synthesis

$$\tilde{u}_0(\psi, y, \varepsilon) = U(\psi, y, P^0(\varphi, y)),$$

constructed owing to the solution of the averaged system, are quasi-optimal with respect to the perturbed system (3.4) *with estimates* $O(\varepsilon)$.

Properties of the approximated solution of the optimal control problem are studied in detail in [6, 134]; higher approximations are also constructed there.

For consideration of systems in the standard form, the condition of 2π -periodicity with respect to ψ should be replaced by the condition of either periodicity or continuous quasi-periodicity with respect to t.

In [6, 134] numerous examples are studied which illustrate the use of the averaging method to the solution of optimal control problems. Let us give a model example where we will underline characteristic features of the analysis of oscillatory systems. We will limit our considerations by an analysis of a quasi-linear system. Considerably non-linear systems are considered in Section 3.3.

Let the motion equation of a controlled system with one degree of freedom be reduced to the form

$$\ddot{x} + \Omega^2 x + \varepsilon f(x, \dot{x}) = \varepsilon u, \qquad x(0) = 0, \qquad \dot{x}(0) = 0, \tag{3.42}$$

where u is a control, a function f accounts for an influence of additional non-linear and non-conservative links, ε is a small parameter. Let further a self-sustained oscillatory regime with given properties be realized in an uncontrolled system (for $u = 0$) (the problem of formation of such a regime is solved in Section 3.2). Let us construct a control providing a transition to a given trajectory in the minimal time for a condition $0 \le u \le U_0$.

Write the main equations and the functional of the problem. By a replacement of variables

$$x = y\cos\psi, \qquad \dot{x} = -\Omega y \sin\psi, \tag{3.43}$$

Eq. (3.42) is reduced to the form

$$\begin{aligned} \dot{y} &= \varepsilon\Omega^{-1}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\sin\psi, \\ \dot{\psi} &= \Omega + \varepsilon(\Omega y)^{-1}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\cos\psi \\ y(0) &= 0, \qquad \psi(0) = \psi_0. \end{aligned} \tag{3.44}$$

The system (3.44) is reduced to the standard form by a replacement $\psi = \Omega t + \theta$. However, owing to the fact that time t is not directly included into coefficients of the system, it is expedient to choose the phase ψ as an independent 'fast' parameter. In this case the order of the equations of the maximum principle decreases.

With an accuracy up to small values $O(\varepsilon^2)$

$$\begin{aligned} \frac{dy}{d\psi} &= \varepsilon\Omega^{-2}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\sin\psi, \\ \frac{dt}{d\psi} &= \Omega^{-1}\left\{1 - \varepsilon(\Omega^2 y)^{-1}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\cos\psi\right\}. \end{aligned} \tag{3.45}$$

Let for $u = 0$ the shortened system

$$\frac{dy}{d\psi} = \varepsilon\Omega^{-2} f^0(y), \tag{3.46}$$

where

$$f^0(y)=\frac{1}{2\pi}\int_0^{2\pi} f(y\cos\psi,-\Omega y\sin\psi)\sin\psi d\psi\,, \tag{3.47}$$

have a unique stable solution $\bar{y}=a$, i.e.,

$$f^0(a)=0\,,\qquad f_y^0(y)\big|_{y=a}<0\,. \tag{3.48}$$

Thus, the control u should transfer the system (3.44) from the state $y(0)=0$ into the state $y(t_f)=a$ in the shortest time.

Recall that a precise periodic (with respect to ψ) solution of the system (3.45) (for $u=0$) is expressed in the form [102]

$$y(\psi,\varepsilon)=a+\varepsilon y_1(a,\psi,\varepsilon)\,, \tag{3.49}$$

where $y_1(a,\psi,\varepsilon)$ is a 2π-periodic component. Hence, it is senseless to fix strictly the boundary condition $y(t_f)=a$; it is sufficient to demand $y(t_f)\in a_\varepsilon$, where a_ε is an ε-zone of the point a. From Eq. $(3.45)_2$ follows that the problem's functional has the form

$$t_f=\Omega^{-1}(\psi_f-\psi_0)+O(\varepsilon)\,. \tag{3.50}$$

Introducing a 'slow' variable $\varphi=\varepsilon\psi$ into consideration, we will re-write the motion equation and functional of the problem in the form

$$\frac{dy}{d\varphi}=\Omega^{-2}\left[f(y\cos\varphi/\varepsilon,-\Omega y\sin\varphi/\varepsilon)-u\right]\sin\varphi/\varepsilon\,, \tag{3.51}$$

$$y(\varphi_0)=0\,,\qquad y(\varphi_f)=a+\varepsilon\rho=a_\varepsilon\,,$$

$$\Phi(u)=\Omega^{-1}\int_{\varphi_0}^{\varphi_f} d\varphi=\min\,. \tag{3.52}$$

Writing the Hamilton's function (3.37)

$$H_1(\varphi/\varepsilon,y,u,p)=\Omega^{-1}\left\{p\Omega^{-1}\left[f(y\cos\varphi/\varepsilon,-\Omega y\sin\varphi/\varepsilon)-u\right]\sin\varphi/\varepsilon-1\right\}, \tag{3.53}$$

we get

$$u=\arg\max_{0\le u\le U_0} H_1(\varphi/\varepsilon,y,u,p)=U_0\eta(p\sin\varphi/\varepsilon)\,, \tag{3.54}$$

where $\eta(x)$ is a unit Heaviside function, p is a Lagrange multiplier satisfying Eq. (3.16).

Let us construct a shortened system (3.22) which determines relations for y^0,

p^0. Substituting (3.54) into (3.53) and averaging over the 'fast' variable, we get

$$H_1(y^0,p^0)=\Omega^{-1}\left\{p^0\left[\Omega^{-1}f^0(y^0)+\frac{U_0}{4}\right]-1\right\}. \tag{3.55}$$

Then, owing to (3.22),

$$\frac{dy^0}{d\varphi}=\frac{1}{\Omega^2}f^0(y^0)+\frac{U_0}{4\Omega}, \quad \frac{dp^0}{d\varphi}=-\frac{1}{\Omega^2}p^0 f_y^0(y^0), \tag{3.56}$$

$$y^0(\varphi_0)=0, \quad y^0(\varphi_f)=a. \tag{3.57}$$

Eq. $(3.57)_1$ can be solved independently of $(3.57)_2$; the phase φ_f of the process end is determined from the boundary condition (3.57). Eq. (3.24) can serve for fixing of the boundary condition $p^0(\varphi)$. The moment of the process end is t_f = $\Omega^{-1}\varepsilon^{-1}(\varphi_f-\varphi_0)$.

3.2 Periodic Control

In Section 3.1 it was shown that the weak control inconsiderably influences the character of the system motion on time intervals $\Delta t \propto 2\pi/\omega$. The case of a control of the periodic (stationary) movement is principally different: the stationary motion does not depend on initial conditions and is determined by the character of acting forces, i.e., by control properties.

Consider, for instance, a problem of formation of a self-sustained oscillatory regime with given properties.

Let dynamics of the controlled system be described by the equation

$$\ddot{x}+\Omega^2 x+\varepsilon f(x,\dot{x})=\varepsilon u, \tag{3.58}$$

where u is a control, f is a function accounting for an influence of non-linear and non-conservative links, ε is a small parameter, $u \in U$.

In a control of resonant systems, a realization of the self-sustained oscillatory regime with a maximal amplitude, or with a fixed amplitude but minimal period, is of main interest. Write the main equations and the functional of the problem. By a replacement of variables

$$x=y\cos\psi, \quad \dot{x}=\Omega y\sin\psi, \tag{3.59}$$

Eq. (3.58) is reduced to the form

$$\dot{y}=\varepsilon\Omega^{-1}\left[f(y\cos\psi,-\Omega y\sin\psi)-u\right]\sin\psi, \tag{3.60}$$

$$\dot{\psi} = \Omega - \varepsilon(\Omega y)^{-1}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\cos\psi\,,$$

or, with an accuracy up to small values $O(\varepsilon^2)$

$$\frac{dy}{d\psi} = \varepsilon\Omega^{-2}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\sin\psi\,,$$

$$\frac{dt}{d\psi} = \Omega^{-1}\left\{1 + \varepsilon(\Omega^2 y)^{-1}\left[f(y\cos\psi, -\Omega y\sin\psi) - u\right]\cos\psi\right\}. \tag{3.61}$$

Let for $u = 0$ the shortened system

$$\frac{d\bar{y}}{d\psi} = \varepsilon\Omega^{-2} f^0(\bar{y})\,,$$

where

$$f^0(y) = \frac{1}{2\pi}\int_0^{2\pi} f(y\cos\psi, -\Omega y\sin\psi)\sin\psi d\psi\,, \tag{3.62}$$

have a unique stable solution $\bar{y} = 0$, i.e.,

$$f^0(0) = 0\,, \qquad f_y^0(0)\Big|_{y=a} < 0\,.$$

Thus, a periodic oscillatory regime can not be realized in the uncontrolled system; existence conditions for such a regime are determined by properties of controlling excitation.

Let us examine some of the most frequent problems of the periodic control.

1. *A self-sustained oscillatory regime with a maximal amplitude.* Let $u = u(\psi, y)$ be some 2π-periodic control realizing a self-sustained oscillatory regime. A periodic solution of Eq. (3.61) has the form [102]

$$y(\psi, \varepsilon) = \bar{y} + \varepsilon y_1(\psi, \bar{y}, \varepsilon)\,, \tag{3.63}$$

where $\bar{y}$ is a constant, $y_1(\psi, \bar{y}, \varepsilon)$ is a 2π-periodic component. Obviously, the problem of a formation problem for a periodic control with a maximal (or fixed) amplitude should be related only to the constant component. It is expedient to express a functional of the problem in the form

$$j_1 = \frac{1}{2\pi}\int_0^{2\pi} y^2(\psi)d\psi = \bar{y}^2 + O(\varepsilon^2)\,. \tag{3.64}$$

Accounting for the relation $d\bar{y}/d\psi = 0$ for a stationary point, we will write the periodicity condition in the form of an integral relation connecting $\bar{y}$ and u:

$$j_2 = \frac{1}{2\pi}\int_0^{2\pi}\left[f(\bar{y}\cos\psi, -\Omega\bar{y}\sin\psi) - u\right]\sin\psi d\psi = 0\,. \tag{3.65}$$

Finally, we can demand that a period of realized oscillations $T(\bar{y})$ should satisfy some additional conditions: either $T(\bar{y}) = T_0$, or, $T(\bar{y}) = \min$. These conditions can be also expressed in the integral form

$$j_3 = \frac{1}{\Omega^2 \bar{y}} \int_0^{2\pi} \left[f(\bar{y}\cos\psi, -\Omega\bar{y}\sin\psi) - u \right] \cos\psi d\psi , \tag{3.66}$$

and

$$j_3 = \Omega T_0 - 2\pi , \tag{3.67}$$

or,

$$j_3 = \min | u \in U . \tag{3.68}$$

2. Let us construct a control forming a periodic regime with given boundary conditions. Let

$$x(0) = \Delta , \quad x(t_f) = -\Delta , \tag{3.69}$$

$$\dot{x}(0) = 0 , \quad \dot{x}(t_f) = 0 , \tag{3.70}$$

where t_f is a fixed (or chosen owing to optimality conditions) moment of the process end.

It follows from (3.69) that a phase of the process end $\psi(t_f) = \pi$ and $\psi(0) = 0$. In this case the condition (3.70) is fulfilled automatically. An existence condition for a periodic regime keeps its form (3.65) for $\bar{y} = \Delta$, and the condition

$$j_4 = (\Omega^2 \Delta)^{-1} \int_0^{\pi} \left[f(\Delta\cos\psi, -\Omega\Delta\sin\psi) - u \right] \cos\psi d\psi , \tag{3.71}$$

expresses constraints imposed on the moment of the process end. If the moment t_f is fixed, then

$$j_4 = \Omega t_f - \pi ; \tag{3.72}$$

if the problem of the optimal high-speed action is formulated then $j_4 = \min$.

Now we will get strict conditions for the maximum principle. Let us show that in the first approximation a solution of the optimal control problem is reduced to the search of stationary points of the averaged system of equations of the maximum principle.

Consider equations of the more general form

$$\begin{aligned} \dot{y} &= \varepsilon f_1(y, \psi, u) , \\ \dot{\psi} &= \omega(y) + \varepsilon f_2(y, \psi, u) , \end{aligned} \tag{3.73}$$

where functions f_1 and f_2 are 2π-periodic with respect to ψ and satisfy conditions of Theorem 3.1, $\omega(y) \geq \omega_0 > 0$. It is also expedient to treat a fast phase ψ in periodic control problems as an independent parameter and to reduce the system (3.73) to the form

$$\frac{dy}{d\psi} = \varepsilon\omega^{-1}(y) f_1(\psi, y, u) + O(\varepsilon^2). \tag{3.74}$$

Then the oscillation period with respect to ψ is fixed: $T_\psi = 2\pi$, and the problem is reduced to the construction of the control $u \in U$ realizing a 2π-periodic regime

$$y(0) = y(2\pi)$$

and minimizing the functional

$$\Phi = \frac{1}{2\pi}\int_0^{2\pi} g(\psi, y, u)\,d\psi\,. \tag{3.75}$$

A function g is considered to be 2π-periodic and piecewise continuous with respect to ψ and sufficiently smooth with respect to y, u. The Hamilton's function of the problem under study has the form

$$H_1(\psi, y, p, q) = -g(\psi, y, u) + \varepsilon\left(q, \omega^{-1}(y) f_1(\psi, y, u)\right). \tag{3.76}$$

Introduce a new adjoint variable $p = \varepsilon q$. It follows from (3.76) that the variable p satisfies the equation

$$\frac{dp}{d\psi} = \varepsilon\frac{dq}{d\psi} = -\varepsilon\frac{\partial H_1}{\partial y},$$

i.e.,

$$\frac{dp}{d\psi} = \varepsilon\frac{dg}{d\psi} - \frac{\varepsilon}{\omega(y)}\left(\frac{\partial H}{\partial y} - \frac{\omega'}{\omega} f_1\right), \tag{3.77}$$

and periodicity conditions $p(0) = p(2\pi)$. Here $\omega' = \omega_y(y)$, $H(\psi, y, u, p) = (p, f_1)$.

Suppose that the optimal control can be uniquely determined form conditions of the maximum principle

$$u = \arg\max_{u \in U}\left[-g(\psi, y, u) + H(\psi, y, u, p)\right] = U(\psi, y, p). \tag{3.78}$$

Substituting (3.78) into (3.77), (3.74) and averaging, we get

$$\frac{dy^0}{d\varphi} = \omega^{-1}(y^0) f^0(y^0, p^0), \qquad \varphi = \varepsilon\psi\,, \tag{3.79}$$

$$\frac{dp^0}{d\varphi}=\frac{\partial g^0\left(y^0,p^0\right)}{\partial y^0}-\frac{\partial}{\partial y^0}\left[\omega^{-1}\left(y^0\right)H^0\left(y^0,p^0\right)\right],$$

where functions f^0, H^0 are determined by Eq. (3.41) and

$$g^0(y,p)=\frac{1}{2\pi}\int_0^{2\pi} g^u(\psi,y,p)d\psi\,,\qquad g^u=g\left(\psi,y,U(\psi,y,p)\right). \tag{3.80}$$

A stationary solution $\bar{y}$, $\bar{p}$ of Eq. (3.79) corresponds to the periodic solution y_T, p_T of the initial system (3.74), (3.77). If coefficients of the system (3.79) satisfy conditions of Theorem A.3, then estimates

$$\left|y_T-\bar{y}\right|\le C\varepsilon\,,\qquad \left|p_T-\bar{p}\right|\le C\varepsilon \tag{3.81}$$

hold for $-\infty<\varphi<\infty$. The period of oscillations is determined by the formula

$$T_0=\int_0^{2\pi}\omega^{-1}(\bar{y})d\psi=\frac{2\pi}{\omega(\bar{y})}. \tag{3.82}$$

As in Section 3.1, it can be shown that the control

$$u_0(\psi)=U(\psi,\bar{y},\bar{p}) \tag{3.83}$$

is quasi-optimal with respect to the initial system with estimates $O(\varepsilon)$. The function (3.83) together with Eq. (3.79) can be treated as the control synthesis law. Linking stationary solutions of the system (3.79) by means of the relation $\bar{p}=\bar{P}(\bar{y})$ following from Eq. (3.79), and considering in (3.83)

$$u(\psi,y)=U\left(\psi,y,P(y)\right), \tag{3.84}$$

we get the control synthesis law.

Obviously, the optimal periodic regime should be stable, in the opposite case an introduction of the control is senseless. Owing to Theorem A.3, a periodic regime y_T is stable if all the eigenvalues of the matrix

$$A(\bar{y})=\bar{f}_y(\bar{y})\,, \tag{3.85}$$

where

$$\bar{f}(y)=\frac{1}{2\pi}\int_0^{2\pi} f\left(\psi,y,u_0\left(\psi,y,P(y)\right)\right)d\psi\,, \tag{3.86}$$

are in the left-hand half-plane (condition A.1). Condition A.1 is stronger than Condition A of Theorem A.8, which provides an existence of a periodic, but not necessarily stable, regime.

Thus, a procedure of the optimal periodic regime construction leads to the following result:

Theorem 3.2. *Let on a domain*

$$D:\quad y \in Y, \quad p \in P, \quad u \in U, \quad -\infty < \psi < \infty,$$

1) *functions* f_1, f_2, ω *satisfy conditions* 1) – 4) *of* Theorem 3.1; *a function* g *satisfies the same conditions*;

2) *the system* (3.79) *has a unique stationary solution* $\bar{y}$, $\bar{p}$ *and satisfies conditions of the second Bogolyubov's theorem* (Theorem A.3);

3) *the function* $f(\psi, y, p)$ *for* $p = P(y)$ *satisfies the condition* A.1.

Then

1. *A periodic problem of the maximum principle* (3.74), (3.77), (3.78) *has a unique solution* $y_T(\psi,\varepsilon)$, $p_T(\psi,\varepsilon)$ *in the* ε*-zone of the stationary point* $\bar{y}$, $\bar{p}$ *of the averaged system* (3.79).

2. *The control*

$$u_0(\psi, y) = U(\psi, y, \bar{p})$$

and the control synthesis

$$u_0(\psi, y) = U(\psi, y, \bar{P}(y)),$$

constructed owing to the solution of the averaged system, are quasi-optimal with respect to the initial system with estimates $O(\varepsilon)$.

3. *The periodic regime under study is stable.*

Let us construct a control synthesis realizing a self-sustained oscillatory regime with a maximal amplitude and a minimal oscillation period. Such a formulation is characteristic for a set of applied problems of formation of the most economical working cycle. We will limit our consideration to a linear system control

$$\ddot{x} + 2\varepsilon\delta\dot{x} + \Omega^2 x = \varepsilon u, \quad |u| \le U_0. \tag{3.87}$$

Here $f(x,\dot{x}) = 2\delta\dot{x}$. Introducing variables 'amplitude – phase' according to Eq. (3.59), we get from (3.61), (3.87)

$$\frac{dy}{d\psi} = -\varepsilon\Omega^{-2}(2\delta y \sin\psi + u)\sin\psi,$$

$$\frac{dt}{d\psi} = \Omega^{-1}\left[1 - \varepsilon(\Omega^2 y)^{-1}(2\delta\Omega y \sin\psi + u)\cos\psi\right]. \tag{3.88}$$

Write the functional of the problem [comp. (3.64), (3.66)]

$$\Phi(u) = \varepsilon\Omega^{-2}\int_0^{2\pi}\left[\lambda y^2 + y^{-1}(2\delta\Omega y \sin\psi + u)\cos\psi\right]d\psi. \tag{3.89}$$

The first term in the functional corresponds to the demand of a maximal amplitude, the second term - to the demand of a minimal period; λ is a constant multiplier. The Hamilton's function of the problem (3.88), (3.89) gets the form

$$H = \varepsilon\left[\lambda y^2 + \left(y^{-1}\cos\psi - p\sin\psi\right)\left(2\delta\Omega y\sin\psi + u\right)\right]\Omega^{-2}, \tag{3.90}$$

where the Lagrange multiplier should be a periodic solution of the equation

$$\frac{dp}{d\psi} = -\frac{\partial H}{\partial y} = \varepsilon\Omega^{-2}\left[y^{-2}u\cos\psi + 2\delta p\Omega\sin^2\psi - 2\lambda y\right]. \tag{3.91}$$

From (3.90), (3.83) we have the following expressions for the optimal control

$$u = -U_0\,\mathrm{sgn}\sin(\psi - \theta) = U(\psi, \bar{y}, \bar{p}), \tag{3.92}$$

where $\cot\theta = \overline{yp}$ and $\bar{y}$, $\bar{p}$ are stationary points of the averaged system corresponding to Eqs. (3.88), (3.91). Substituting (3.92) into (3.88), (3.91) and averaging, we get the equations determining $\bar{y}$ and θ:

$$\frac{2U_0}{\pi\Omega^2}\cos\theta - \frac{\delta}{\Omega}\bar{y} = 0,$$
$$\frac{2U_0}{\pi\bar{y}^2}\sin\theta + \delta\bar{p}\Omega - 2\lambda\bar{y} = 0, \qquad \bar{p} = \cot\theta/\bar{y}. \tag{3.93}$$

If the oscillation amplitude $\bar{y}$ is given then Eq. $(3.93)_1$ serves for a switch–point determination, while Eq. $(3.93)_2$ serves for calculation of the multiplier λ. Let us note the following: it follows from (3.88) that the period change linked with the control introduction is

$$\varepsilon\Delta T = T - T_0 = -\varepsilon\Omega^{-3}\bar{y}^{-1}\int_0^{2\pi} u\cos\psi\,d\psi = \varepsilon\frac{4U_0\sin\theta}{\Omega^3\bar{y}}, \tag{3.94}$$
$$T_0 = 2\pi/\Omega.$$

In its turn, the oscillation amplitude

$$\bar{y} = \frac{2U_0\cos\theta}{\pi\Omega\delta}, \tag{3.95}$$

i.e.,

$$\Delta T = 2\tan\theta\pi\delta/\Omega^2. \tag{3.96}$$

Thus, the oscillation amplitude depends neither on the switch moment nor on the control level, and the period change is determined only by the choice of the switch point. In particular, the maximal amplitude is reached for $\theta = 0$: $y_{\max} = 2U_0 =$ $2U_0/\pi\Omega\delta$ (for $\Delta T = 0$).

Accounting for (3.59), Eq. (3.92) can be expressed in the form

$$u = U_0\,\mathrm{sgn}\left[\dot{x}\Omega^{-1}\cos\theta + x\sin\theta\right]. \tag{3.97}$$

In particular, the optimal control for $\theta = 0$ has the form $u = U_0 \operatorname{sgn} \dot{x}$.

3.3 Processes of Oscillation Settlement in Vibroimpact Systems

The method of integral equations given in Section 2 serves for analysis of periodic motions of vibroimpact systems. The main method of analysis of transition processes was an adding method, or the method of point mapping. But even for the simplest systems, linear between impacts, it was possible to find only numerical solutions. More complications appear in solution of optimization problems, where initial conditions, impact conditions and transversal conditions should be accounted for.

Another approach based on the transformation of motion equations with the help of a non-smooth replacement of variables was developed in [20, 33, 57]. These transformations allow the reduction of motion equations of the system, close to conservative, to the standard form and the use of the averaging method.

An analogous approach is also effective for solution of optimal control problems: a system is reduced to the standard form, and the problem is solved in the same way as for smooth systems.

In Section 3.3.1 a procedure of the reduction of motion equations of a vibroimpact system to the standard form by means of replacement of variables suggested in [20] is given. Further, in Section 3.3.2, control problems for oscillation settlement are considered. In Section 3.3.3 the averaging method is used for solution of periodic control problems. In Sections 3.3.4 and 3.3.5 some particular problems are examined.

3.3.1 General Equations and Replacement of Variables

Consider a quasi-conservative vibroimpact system with one degree of freedom, the motion of which is described by the equation

$$\ddot{x} + \Omega^2 x = \varepsilon g(t, x, \dot{x}), \quad x(0) = \Delta, \quad \dot{x}_+(0) = -v_0, \tag{3.98}$$

and by the condition of an impact against a one-side limiter

$$x = \Delta, \quad \dot{x}_+ = -(1 - \varepsilon r)\dot{x}_-. \tag{3.99}$$

Here $g(t, x, \dot{x})$ is a piecewise continuous function of its arguments characterizing additional non-linear and non-conservative forces, ε is a small parameter. For $\varepsilon = 0$ the generating system

$$\ddot{x} + \Omega^2 x = 0, \tag{3.100}$$

$$x = \Delta\,, \qquad \dot{x}_+ = -\dot{x}_-\,,$$

is conservative and has the common integral (3.60), corresponding to a periodic one-impact regime

$$x(t) = -J\chi(\psi,\omega)\,, \tag{3.101}$$

where

$$\psi = \omega(J)(t - t_0) \tag{3.102}$$

and $\chi(\psi,\omega)$ is a periodic Green's function of the first kind (index 1 is everywhere omitted), periodic with respect to ψ with the period 2π

$$\chi(\psi,\omega) = \frac{1}{2\pi\omega}\sum_{k=-\infty}^{\infty}\frac{\exp ki\psi}{(\Omega/\omega)^2 - k^2}\,, \tag{3.103}$$

or, in a closed form

$$\chi(\psi,\omega) = \frac{1}{2\Omega}\frac{\cos(\Omega/\omega)(\psi - \pi)}{\sin(\pi\Omega/\omega)}\,, \qquad 0 \le \psi \le 2\pi\,. \tag{3.104}$$

Arbitrary constants J, t_0 are determined by initial conditions $t = t_0$, $J = J_0$; a function $\omega(J)$ is given by an impact condition

$$x(t_0) = \Delta\,, \qquad J = -2\Omega\tan(\pi\Omega/\omega)\,. \tag{3.105}$$

A condition of the velocity discontinuity is fulfilled automatically, since

$$\dot{x} = -\omega J\chi_\psi(\psi,\omega)\,, \tag{3.106}$$

and

$$\chi_\psi(+0) = -\chi_\psi(-0) = 1/(2\omega)\,. \tag{3.107}$$

Suppose that small disturbances do not change a qualitative motion character and regime remains close to one-impact periodic one. Then a method of variation of arbitrary variables in systems close to conservative ones can be used for a construction of a general solution of Eqs. (3.98), (3.99).

Introduce new variables 'impulse – phase' with the help of the relations

$$x = -J\chi(\psi,\omega)\,, \qquad \dot{x} = -\omega J\chi_\psi(\psi,\omega)\,, \tag{3.108}$$

where it is assumed that J, ψ are functions of time, and the dependence $\omega(J)$ is determined by relation (3.105). Differentiating x we get a compatibility condition

$$\dot{x} = \dot{J}(J\chi)_J - (J\chi_\psi)\dot{\psi} = -\omega J\chi_\psi\,, \tag{3.109}$$

where the dependence $\omega(J)$ is accounted for in differentiating of χ_J: $x_J = \chi_\omega \omega_J$. Further, substituting (3.108) into (3.98), we obtain

$$\ddot{x} = -\omega J \chi_{\psi\psi} \dot{\psi} - \omega\left(J\chi_\psi\right)_J \dot{J} = \Omega^2 J\chi + \varepsilon g\left(t, -J\chi, -J\omega\chi_\psi\right). \tag{3.110}$$

In periods between impacts $2\pi k < \psi < 2\pi(k+1)$, $k = 0, \ldots$ Let us single out the system solved with respect to $\dot{J}$, $\dot{\psi}$ from (3.109), (3.110). Taking into account that an equality

$$J\omega^2 \chi_{\psi\psi} + \Omega^2 J\chi = 0$$

identically holds, owing to (3.100), (3.101), we get analogously [20]

$$\dot{J} = d^{-1} g\left(t, -J\chi, -J\omega\chi_\psi\right)\left(J\chi_\psi\right), \quad J(0) = (2 - \varepsilon r)v_0 = \xi,$$
$$\dot{\psi} = \omega(J) - \varepsilon d^{-1} g\left(t, -J\chi, -J\omega\chi_\psi\right)(J\chi)_J, \quad \psi(0) = 0,$$

where, with account for (3.104) – (3.110),

$$d = \omega J\left[\chi_{\psi\psi}(J\chi)_J - \chi_\psi\left(J\chi_\psi\right)_J\right] = -\frac{J}{4\omega},$$

i.e., in periods between impacts the system motion is described by equations

$$\dot{J} = -4\varepsilon g\left(t, -J\chi, -J\omega\chi_\psi\right)\omega\chi_\psi,$$
$$\dot{\psi} = \omega(J) + 4\varepsilon\omega J^{-1} g\left(t, -J\chi, -J\omega\chi_\psi\right)(J\chi)_J. \tag{3.111}$$

The first of the impact conditions (3.99) is accounted for in an assumption that the function $\omega(J)$ keeps the form (3.105), the velocity discontinuity condition with account for (3.107) is reduced to the form

$$\psi = 2\pi k, \quad J_+^k = (1 - \varepsilon r)J_-^k, \quad k = 0, 1, \ldots, \tag{3.112}$$

where

$$J_\pm^k = J\big|_{\psi = 2\pi k \pm 0},$$

or,

$$\Delta J_k = J_+^k - J_-^k = -\varepsilon r J_-^k, \quad \psi = 2\pi k. \tag{3.113}$$

Impact moments t_k are determined by the equality $\psi(t_k) = 2\pi k$.

Accounting for properties of the delta function and substituting (3.113) into motion equations, we obtain

$$\dot{J} = -\varepsilon\omega\left[r\delta_-^{2\pi}(\psi)J + 4g\left(t, -J\chi, -J\omega\chi_\psi\right)\chi_\psi\right], \tag{3.114}$$

$$J(0)=\xi .$$

Here

$$\delta_-^{2\pi}(\psi)J = \sum_{k=-\infty}^{\infty}\delta(\psi - 2\pi k)J_-^k(\psi),$$

Eq. $(3.111)_2$ remains without changes.

The work [20] is dedicated to the analysis of Eqs. (3.111), (3.114). A method of averaging expanded in [33, 57, 117] to systems with discontinuity of coordinates can be used for their solution.

Let $g = g(x,\dot{x})$. Then a solution of the system (3.111), (3.114) is approximated by a solution of shortened equations

$$\dot{J}^0 = -\varepsilon\left[rJ^0T^{-1} + 4\omega g_1(J^0)\right], \quad J^0(0) = 2v_0 = \xi, \tag{3.115}$$

$$\dot{\psi} = \omega(J^0)\left[1 + 4\varepsilon g_2(J^0)\right], \tag{3.116}$$

being obtained by averaging of (3.111), (3.114) with an account for a 2π-periodicity with respect to ψ of the right-hand part of equations. Here $T = T(J) = 2\pi/\omega(J)$,

$$g_1(J) = \frac{1}{2\pi}\int_0^{2\pi} g(-J\chi, -J\omega\chi_\psi)\chi_\psi d\psi,$$

$$g_2(J) = \frac{1}{2\pi J}\int_0^{2\pi} g(-J\chi, -J\omega\chi_\psi)(J\chi)_J d\psi, \tag{3.117}$$

and the estimate [34]

$$\left|J(\tau) - J^0(\tau)\right| \le C\varepsilon, \quad 0 \le \tau \le L \quad (\text{or } 0 \le \tau < \infty) \tag{3.118}$$

holds. Here and below C are constants which do not depend on ε.

A strict substantiation of the averaging method with discontinuous and impulse right-hand parts is in [33]. For fulfillment of a transition from Eqs. (3.111), (3.114) to the averaged equations (3.115), (3.116) with an error (3.118), it is enough to demand that in the domain $J \in K$ ($|J| \le k$), $\psi \in R_1$ the following conditions hold [33]

a) functions $g(-J\chi, -J\omega\chi_\psi)$, $J^{-1}g(-J\chi, -J\omega\chi_\psi)$ are continuous with respect to J and bounded with respect to ψ continuously with respect to other variables;

b) mean values (3.117) and the frequency $\omega(J) \ge \omega_0 > 0$ satisfy the Lipschitz condition with respect to J;

c) Eq. (3.115) has an asymptotically stable solution $J^0(\tau)$ which belongs to the domain K together with its ρ-zone.

[In a case of fulfillment of conditions a) and b) the estimate (3.118) holds true for a finite time interval; if additionally the condition c) is satisfied then the estimate (3.118) holds for $0 \le \tau < \infty$]

The asymptotically stable stationary solution of Eq. (3.115) determined by the condition

$$\dot{J}^0 = 0, \quad rJ^0 = 2\pi g_1(J^0), \tag{3.119}$$

fixes parameters of self-sustained oscillations with a period $T_0 = 2\pi/\omega(J_0)$.

A case of a quasi-isochronous system is of special interest. If a clearance is small, $\Delta = \varepsilon\Delta_1$, then, using a replacement

$$x_1 = x - \varepsilon\Delta_1, \tag{3.120}$$

we can reduce Eq. (3.98) and the impact condition to the form

$$\ddot{x}_1 + \Omega^2 x_1 = \varepsilon\left[g(x_1 + \varepsilon\Delta_1, \dot{x}_1) - \Omega^2\Delta_1\right], \tag{3.121}$$
$$x_1 = 0, \quad \dot{x}_{1+} = -(1-\varepsilon r)\dot{x}_{1-}$$

(for the sake of simplicity the system is considered to be autonomous). For $\Delta = 0$ the generating system

$$\ddot{x}_1 + \Omega^2 x_1 = 0, \quad x_1 = 0, \quad \dot{x}_{1+} = -\dot{x}_{1-}$$

has a constant eigenfrequency $\omega = \omega_0 = 2\Omega$, and the system under study is quasi-isochronous. By a substitution

$$x_1 = -J\chi(\psi, \omega_0), \quad \dot{x}_1 = -\omega_0 J\chi_\psi(\psi, \omega_0) \tag{3.122}$$

Eq. (3.121) is reduced to the form analogous (3.111), but $\omega_J \equiv 0$ and $(J\chi)_J = \chi$, i.e.,

$$\begin{aligned} \dot{J} &= -4\varepsilon\omega_0\left[g(-J\chi + \varepsilon\Delta_1, -J\omega_0\chi_\psi) - \Omega^2\Delta_1\right]\chi_\psi, \\ \dot{\psi} &= \omega_0 + 4\varepsilon\omega_0 J^{-1}\left[g(-J\chi + \varepsilon\Delta_1, -J\omega_0\chi_\psi) - \Omega^2\Delta_1\right]\chi. \end{aligned} \tag{3.123}$$

The shortened system with the terms up to $O(\varepsilon)$ has the form

$$\begin{aligned} \dot{J}^0 &= \varepsilon\left[-rJ^0T_0^{-1} - 4\omega_0 g_1(J^0)\right], \quad T_0 = \pi/\Omega, \\ \dot{\psi} &= \omega_0 + 4\varepsilon\omega_0\left[g_2(J) - \gamma\Omega^2\Delta_1\right], \\ \gamma &= \frac{1}{2\pi}\int_0^{2\pi}\chi(\psi, \omega_0)d\omega, \end{aligned} \tag{3.124}$$

i.e., the impulse in the first approximation does not depend on the clearance in the autonomous quasi-isochronous system. A suggestion about the quasi-isochronous character (i.e., the smallness of the clearance) considerably simplifies the computational procedure not violating the qualitative notions on the motion character of the system, so for practical application it is always considered that $\Delta = \varepsilon\Delta_1$.

3.3.2 Main Equations of Motion Control

Let the system motion during periods between impacts be described by the equation

$$\ddot{x} + \Omega^2 x = \varepsilon g(x, \dot{x}, u) \tag{3.125}$$

and by conditions of the impact against the one-side limiter (3.99). Suppose that the generating solution describes a one-impact T-periodic regime. A control introduction, generally speaking, can excite another kinds of regimes different from one-impact. It was shown in [62] that the maximal impulse for a case of a stationary motion is realized just in the one-impact regime. So, we are looking for a control when the one-impact character of the regime remains.

The replacement (3.108) reduces (3.125) to the system of equations in the standard form with a fast rotating phase

$$\begin{aligned} \dot{J} &= -4\varepsilon\omega g(-J\chi, -J\omega\chi_\psi, u)\chi_\psi, \\ \dot{\psi} &= \omega\left[1 + 4\varepsilon J^{-1} g(-J\chi, -J\omega\chi_\psi, u)(J\chi)_J\right], \end{aligned} \tag{3.126}$$

supplemented by the discontinuity conditions (3.113). It is supposed that for any admissible control the right-hand parts of Eqs. (3.126) satisfy conditions providing applicability of the averaging method.

Consider some problems of the optimal control of vibroimpact systems described by Eqs. (3.126). It is supposed that the solution of the problems under study exists and can be determined as a unique solution of the system of equations of the maximum principle constructed with an account for discontinuity conditions with respect to the variable J. We will use the averaging procedure developed in Sections 3.1, 3.2.

The problem of an optimal high-speed action. We will construct a control $u(t)$, $|u| \le U_0$, which transfers the system from its initial position with an impact impulse v_0 to a final one, with an impulse v_* corresponding the impulse of the chosen working regime in a shortest time $\Delta t = t_*$. Strictly speaking, it should be requested that the final impulse $I_n = (1+R)\dot{x}_-$ at the moment of the nth impact,

$\psi = 2\pi n$, should satisfy the condition $I_n = v_*$, and thus determine a duration of an acceleration process. At the same time, from (3.108), (3.113) follows that for $\psi = 2\pi n$ the relation

$$I_n = \frac{1}{2}\left(J_-^n + J_+^n\right) \tag{3.127}$$

holds. In its turn, from (3.123), (3.127) we have

$$\left|I_n - J_-^n\right| \le C\varepsilon, \quad \left|I_n - J_+^n\right| \le C\varepsilon, \tag{3.128}$$

i.e., boundary conditions with respect to variable J can be introduced with a given accuracy in the form $J_+(0) = v_0$, $J_-(t_*) = v_*$ not linking the moment of the process end with the impact moment.

Eq. (3.126) and the problem's constraints do not depend directly on time, so it is expedient to treat ψ as a new additional variable and to reduce (3.126) to the form

$$\frac{dJ}{d\psi} = -4\varepsilon g\left(-J\chi, -J\omega\chi_\psi, u\right)\chi_\psi + \varepsilon^2 \rho(J, \psi, u). \tag{3.129}$$

Here ρ is a rest term, continuously bounded in the domain under study.

Since there is a unique linkage between ψ and t

$$t_* = \int_0^{\psi_*} \omega^{-1}\left(J(\psi)\right) d\psi + O(\varepsilon t_*), \tag{3.130}$$

then the problem of the optimal high-speed action is equivalent to the minimization problem for the integral (3.130). In this case, owing to (3.127) – (3.130), an error of the impulse determination is of order ε, while a determination error for the moment t_* is of order εt_* over the interval $t_* \propto L\varepsilon^{-1}$.

Let us construct the Hamiltonian of the problem (3.129), (3.130), (3.113). Owing to the general rules (see Section A.1), we have

$$H = -\varepsilon\left[\omega^{-1}(J) + 4pg\left(-J\chi, -J\omega\chi_\psi, u\right)\right] + O(\varepsilon^2),$$
$$2\pi(k-1) < \psi < 2\pi k, \tag{3.131}$$
$$H_+(2\pi k) = H_-(2\pi k).$$

A small parameter in the first term means that a non-fixed moment of the process end $\psi_* = O(\varepsilon^{-1})$. The adjoint variable p satisfies the condition

$$dp/d\psi = -\partial H/\partial J \tag{3.132}$$

and the discontinuity condition which is adjoint to (3.113)

$$p_- = Rp_+, \quad \psi = 2\pi k, \quad R = 1 - \varepsilon r,$$

i.e.,

$$p_+(2\pi k) - p_-(2\pi k) = \varepsilon r p_+(2\pi k) = \varepsilon r p_+^k. \tag{3.133}$$

Substituting (3.113), (3.133) into Eqs. (3.129), (3.130) and limiting the consideration with the terms up to $O(\varepsilon)$, we obtain

$$\begin{aligned} \frac{dJ}{d\psi} &= -\varepsilon\left[rJ\delta_-^{2\pi}(\psi) + 4g\left(-J\chi, -J\omega\chi_\psi, u\right)\chi_\psi\right], \\ \frac{dp}{d\psi} &= -\varepsilon\left\{rp\delta_+^{2\pi}(\psi) + 4p\left[g\left(-J\chi, -J\omega\chi_\psi, u\right)\chi_\psi\right]_J + \left[\omega^{-1}(J)\right]_J\right\}; \end{aligned} \tag{3.134}$$

boundary conditions are reduced to the form $J(0) = v_0$, $J\left(\varepsilon^{-1}\theta\right) = v_*$, $\theta = O(1)$. Here

$$p\delta_+^{2\pi}(\psi) = \sum_{k=-\infty}^{\infty} \delta(\psi - 2\pi k) p_+(2\pi k).$$

The control u is determined from conditions of the maximum principle

$$u_*(\psi) = \arg\max_{|u| \le U_0} H(\psi, J, u, p) = U\left(\psi, J(\psi), p(\psi)\right). \tag{3.135}$$

Substituting (3.135) into (3.134) and averaging, we get the shortened equations determining the zero-approximation

$$\begin{aligned} \frac{dJ^0}{d\varphi} &= -\frac{r}{2\pi}J^0 - 4g^*\left(J^0, p^0\right), \qquad \varphi = \varepsilon\psi, \\ \frac{dp^0}{d\varphi} &= \frac{r}{2\pi}p^0 + 4p^0 g_J^*\left(J^0, p^0\right) - \frac{\omega_J\left(J^0\right)}{\omega^2\left(J^0\right)}, \\ J^0(0) &= v_0, \quad J^0(\theta) = v_*, \end{aligned} \tag{3.136}$$

where

$$g^*(J, p) = \frac{1}{2\pi}\int_0^{2\pi} g\left(-J\chi, -J\omega\chi_\psi, U(\psi, J, p)\right)\chi_\psi\, d\psi. \tag{3.137}$$

Analysis of Eqs. (3.135), (3.136) is considerably complicated by dependence $\chi(\psi)$ on $\omega(J)$.

Strictly speaking, conclusions of [6] which substantiates an utilization of the averaging method in optimal control problems are inapplicable to Eqs. (3.134), (3.135). In [6] a measurability of right-hand parts of equations in the standard form with respect to a fast variable (see Section A.3) was assumed, as well as the fact that Eq. (3.134) contains delta functions.

In order to prove the closeness of a solution of two-point boundary–value problems (3.134) and (3.136), it is sufficient to repeat all considerations of [6] taking into account for the closeness of the Cauchy problem for an initial and averaged systems [34] (see Section A.3). It is also possible to change the scheme of the approximate solution, preliminarily undertaking a partial averaging in the system (3.129) with discontinuity conditions (3.113), and then to construct an optimal control on trajectories of the obtained partially averaged system, not containing delta functions. Such an approach is developed below in Section 3.3.4.

Consider some particular cases.

a) Let dynamics of a system between impacts be described by the equation

$$\ddot{x} + \Omega^2 x + 2\varepsilon b\dot{x} + \varepsilon g_0(x) = \varepsilon u\,, \tag{3.138}$$

i.e.,

$$g(x,\dot{x},u) = -g_0(x) - 2b\dot{x} + u\,,$$

and, owing to(3.135), (3.137),

$$u_* = -U_0 \operatorname{sgn}(p\chi_\psi), \quad g^* = 2bJ\omega k_2 - U_0 k_1 \operatorname{sgn} p\,, \tag{3.139}$$

where

$$k_1(J) = \frac{1}{2\pi}\int_0^{2\pi} |\chi_\psi| d\psi > 0\,, \quad k_2(J) = \frac{1}{2\pi}\int_0^{2\pi} \chi_\psi^2 d\psi > 0\,.$$

Thus, Eqs. (3.136) will get the form

$$\begin{gathered}\frac{dJ^0}{d\varphi} = Q(J^0, p^0), \quad J^0(0) = v_0\,, \quad J^0(\theta) = v_*\,,\\ \frac{dp^0}{d\varphi} = -p^0 Q_J(J^0, p^0) + \left[\omega^{-1}(J^0)\right]_J\,,\\ Q(J^0, p^0) = -\frac{1}{2\pi} J^0 - 8bJ^0\omega k_2 + 4U_0 k_1 \operatorname{sgn} p^0\,.\end{gathered} \tag{3.140}$$

It is clear from physical considerations, the a demand

$$\frac{dJ^0}{d\varphi} = Q(J^0, p^0) > 0\,, \quad v_0 \le J \le v_*\,, \tag{3.141}$$

corresponds for $v_* > v_0$ to an optimal high-speed action; moreover,

$$4U_0 k_1 > rJ^0/2\pi + 8bJ^0\omega k_2 > 0\,, \tag{3.142}$$

i.e., $\operatorname{sgn} p^0 = 1$. Let us show that a sign–constant solution correspond to Eq. $(3.140)_2$ when condition (3.142) is fulfilled.

Let $Q(J^0) = Q(J^0, p^0)$ for a constant value $\operatorname{sgn} p^0 = 1$. Then, introducing the Hamilton's function of the shortened system (3.140)

$$H^0(J^0, p^0) = -\omega^{-1}(J^0) + p^0 Q(J^0),$$

into consideration and working as in Section 3.1, it can be shown that for the problem of the optimal high-speed action the relation

$$H^0\big|_{\varphi=\varphi_*^0} = -\omega^{-1}(v_*) + p^0(\varphi_*^0)Q(v_*) = 0$$

holds true. Here φ_*^0 is an unknown phase of the process end, determined by the boundary condition $J^0(\varphi_*^0) = v_*$. From the other hand, it follows from the autonomy of the system (3.140) [see (A.22)]

$$H^0(J^0, p^0) = \text{const}(\varphi) = H^0\big|_{\varphi=\varphi_*^0} = 0,$$

i.e., for all φ

$$p^0(\varphi) = \left(\omega(J^0)Q(J^0)\right) > 0.$$

Thus,

$$u_*^0 = -U_0 \operatorname{sgn} \chi_\psi,$$

or, accounting for (3.108),

$$u_*^0 = U_0 \operatorname{sgn} \dot{x}. \tag{3.143}$$

The control (3.143) is quasi-optimal in the mentioned above sense.

The time of high-speed action can be estimated via solution of Eq. (3.141) with an assumption $J^0(\theta) = v_*$. In particular, in a system with a small clearance $\Delta = \varepsilon\Delta_1$, $\omega = \omega_0 = 2\Omega$, we will obtain

$$\frac{dJ^0}{d\varphi} = -\mu(J_0 - \lambda U_0), \tag{3.144}$$

where $\mu = r/2\pi + b/2\Omega > 0$, $\lambda = (\pi\Omega\mu)^{-1}$ and

$$J^0 = (v_0 - \lambda U_0)e^{-\mu\varphi} + \lambda U_0, \tag{3.145}$$

so that

$$e^{\mu\varphi} = \frac{\lambda U_0 - v_0}{\lambda U_0 - v_*} > 1. \tag{3.146}$$

It follows form (3.146) that the acceleration problem has the sense only for

$$U_0 > \frac{\pi\Omega\mu v_*}{2}. \tag{3.147}$$

If $2U_0 = \pi\Omega\mu v_*$ then a regime of self-sustained oscillations with an impulse

$J_0 = v_*$ is being realized in the system, and the acceleration time $\theta = \infty$.

b) *Autoresonant system.* Let dynamics of a one-mass system be described by the equation

$$\ddot{x} + 2\varepsilon b\dot{x} + \Omega^2 x + \varepsilon ux = 0\,, \tag{3.148}$$

i.e.,

$$g = -2b\dot{x} - ux\,,$$

and, owing to(3.140), (3.135), (3.137)

$$\begin{aligned} u_* &= -U_0 \operatorname{sgn} p \operatorname{sgn}[\chi(\varphi)\chi_\psi(\varphi)]\,, \\ g^* &= 2bJ\omega k_2 + JU_0 \operatorname{sgn} pk_3\,, \end{aligned} \tag{3.149}$$

where $k_2(J)$ is the same coefficient as in (3.139),

$$k_3 = \frac{1}{2\pi}\int_0^{2\pi} |\chi(\psi)||\chi_\psi(\psi)|d\psi > 0\,, \qquad k_3 = k_3(J)\,. \tag{3.150}$$

Eqs. (3.137) get the form

$$\begin{aligned} \frac{dJ^0}{d\varphi} &= Q(J^0, p^0)\,, \\ \frac{dp^0}{d\varphi} &= \left[\omega^{-1}(J^0)\right]_J - p^0 Q_J(J^0, p^0)\,, \\ Q(J^0, p^0) &= -J^0\left[r/2\pi + 8b\omega k_2 + 4U_0 k_3 \operatorname{sgn} p^0\right]. \end{aligned} \tag{3.151}$$

Working in the same way as in the previous case, it can be shown that in the acceleration problem, optimal with respect to the high-speed action, it should be considered

$$\begin{aligned} &\operatorname{sgn} p^0 = -1\,, \\ &4U_0 k_3(J^0) > \left[r/2\pi + 8b\omega(J^0)k_2(J^0)\right], \qquad v_0 < J < v_*\,, \end{aligned} \tag{3.152}$$

i.e., owing to (3.108),

$$u_*^0 = -U_0 \operatorname{sgn}(\chi\chi_\psi) = -U_0 \operatorname{sgn}[x\dot{x}]\,. \tag{3.153}$$

Introducing the control (3.153) into the initial system (3.138), we get a autoresonant system providing the acceleration regime, optimal with respect to the high-speed action.

3.3.3 Periodic Control of Quasi-Conservative Systems

A finite value of the impulse $J = v_*$ is determined by parameters of a stationary periodic regime. Let us construct a control $u(t)$, $|u| \le U_1$, forming a maximal impact impulse for a periodic movement of the system (3.125).

The system (3.125) can be reduced to the form (3.126) by means of the replacement (3.108). Considering the control aim to be a supporting of a near-resonant one-impact regime, we will construct a periodic control $u(\psi) = u(\psi + 2\pi)$ and a 2π-periodic solution of Eq. (3.129) corresponding to this control. As follows form (3.107), (3.108), an impulse at the impact moment in a periodic solution

$$\Phi_1 = I = (1+R)\dot{x}_- = (1-\varepsilon r/2)J_+(\psi_k), \quad \psi_k = 2\pi k, \tag{3.154}$$

where $J(\psi)$ is a periodic solution of Eq. (3.126) [or (3.129)]. It follows from (3.118) that a periodic solution of Eq. (3.126) can be presented in the form

$$J(\psi) = \bar{J} + \varepsilon J_1(\bar{J}, \psi, \varepsilon),$$

where $\bar{J}$ is a constant, J_1 is a 2π-periodic component with discontinuities in points $\psi_k = 2\pi k$, $k = 0, \pm 1, \ldots$ Since it is not expedient in a control problem to formulate conditions for a discontinuity point, let us optimize the system (3.129) according to the integral criterion [comp. (3.64)]

$$\Phi(u) = \frac{1}{2\pi}\int_0^{2\pi} J(\psi)\,d\psi. \tag{3.155}$$

The Hamilton's function of the problem (3.129), (3.155) is reduced to [comp. (3.76)]

$$H = J - 4\varepsilon q g(-J\chi, -J\omega\chi_\psi, u)\chi_\psi. \tag{3.156}$$

Working in the same way as in Section 3.2, we obtain that a variable $p = \varepsilon q$ satisfies the equation $dp/d\psi = -\varepsilon \partial H/\partial J$, the discontinuity condition (3.133) and the condition of 2π-periodicity with respect to ψ. Substituting the discontinuity conditions into equations of the maximum principle and leaving the terms $O(\varepsilon)$, we get

$$\frac{dJ}{d\psi} = -\varepsilon\left[rJ\delta_-^{2\pi}(\psi) + 4g(-J\chi, -J\omega\chi_\psi, u)\chi_\psi\right],$$

$$\frac{dp}{d\psi} = \varepsilon\left\{rp\delta_+^{2\pi}(\psi) + 4p\left[g(-J\chi, -J\omega\chi_\psi, u)\chi_\psi\right]_J - 1\right\}, \tag{3.157}$$

where

$$u_*(\psi) = \arg\max_{|u|\le U_1} H(\psi, J, u, p) = U(\psi, J, p). \tag{3.158}$$

Substituting u_* into (3.157) and averaging, we get

$$\begin{aligned} \frac{dJ^0}{d\varphi} &= -\frac{r}{2\pi}J^0 - 4g^*(J^0, p^0), \\ \frac{dp^0}{d\varphi} &= \frac{r}{2\pi}p^0 + 4p^0 g_J^*(J^0, p^0), \end{aligned} \tag{3.159}$$

where g^* is the same function as in (3.137). A stationary solution of the system (3.159)

$$\begin{aligned} &\frac{r}{2\pi}J^0 + 4g^*(J^0, p^0) = 0, \qquad J^0 = \bar{J}, \\ &p^0\left[\frac{r}{2\pi} + 4g_J^*(J^0, p^0)\right] = 1, \qquad p^0 = \bar{p}, \end{aligned} \tag{3.160}$$

and a control

$$u_*^0 = U(\psi, \bar{J}, \bar{p}) \tag{3.161}$$

correspond to the periodic control.

A period of a stationary regime in the first approximation is determined by a relation $T_0 = 2\pi/\omega(\bar{J})$.

Let us construct a periodic control for the system (3.138). From (3.156), (3.158), (3.161) we have

$$u_*^0 = -U_1 \operatorname{sgn}(\bar{p}\chi_\psi), \tag{3.162}$$

and, consequently,

$$\begin{aligned} &\frac{r}{2\pi}\bar{J} + 8b\bar{J}\omega(\bar{J})k_2 = 4U_1 k_1 \operatorname{sgn}\bar{p}, \\ &p\left\{\frac{r}{2\pi} + \left[8b\bar{J}\omega(\bar{J})k_2 - 4U_1 k_1 \operatorname{sgn}\bar{p}\right]_J\right\} = 1, \end{aligned} \tag{3.163}$$

where k_1, k_2 are the same coefficients as in (3.139). It follows from $(3.163)_1$ that $\operatorname{sgn}\bar{p} = 1$, $\bar{p} > 0$. Hence, the quasi-optimal control can be constructed in the form

$$u_*^0 = -U_1 \operatorname{sgn}\chi_\psi = U_1 \operatorname{sgn}\dot{x}. \tag{3.164}$$

We will show that the realized regime is stable. Really, if $\bar{p} > 0$ then a coefficient of $\overline{p}$ in Eq. (3.163)$_2$ is always positive. It is easy to show that a positiveness condition coincides with a condition of asymptotic stability of the stationary solution $\overline{J}$ [20].

Let us determine a linkage between limit possibilities of a control in transition and stationary regimes.

Let J_* be a stationary value of the impulse corresponding to the control (3.164) and determining parameters of an optimal periodic regime. It is obvious that $U_0 > U_1$ should be considered for bringing the system to a working regime from the initial state $J = v_0 < J_*$ with the help of the control (3.143); the capacity supply $\Delta U = U_0 - U_1$ is determined by a necessary velocity of a transition to a regime of self-sustained oscillations. Let us calculate parameters of the stationary regime in a system with a small clearance, for $\omega = 2\Omega$. From (3.163) for $\operatorname{sgn} p = 1$ we have

$$J_* = v_* = \lambda U_1, \tag{3.165}$$

where still $\lambda = 2(\pi\Omega\mu)^{-1}$, $\mu = r/2\pi + b/2\Omega$.

If only the working regime is being optimized, then the control (3.164) is introduced in the system. We will show that for sufficiently small dissipation μ, the regime moves sufficiently close to the self-sustained oscillation one in a finite number of cycles.

Let us limit our considerations to a case of quasi-isochronous system. Let us estimate a time of transition from the initial state $J = v_0 = \kappa_1 J_*$, $\kappa_1 << 1$, to any arbitrarily close zone $J = (1-\kappa_2)J_*$, $\kappa_2 << 1$, of an optimal regime for $u = U_1 \operatorname{sgn} \dot{x}$. A solution is given by Eq. (3.146) with substitution of $U_0 = U_1$, $v_0 = \kappa_1 J_*$, $v_* = J_*(1-\kappa_2)$, $J_* = \lambda U_1$. Then

$$e^{\mu\theta} = (1-\kappa_1)\kappa_2^{-1}.$$

Let $\kappa_1 = \kappa_2 = 0.1$, $\mu = 0.3$. Then $\theta < 4\pi$, i.e., the system practically transits in two cycles into a self-sustained oscillation regime, and the additional optimization with respect to a high-speed action of the transition regime is not necessary.

3.3.4
Partial Averaging

In many application problems it is possible to reduce sufficiently a computational procedure if the initial system (3.126) or (3.129) is replaced by a smooth, partially averaged system.

Let $g(x,\dot{x},u) = g_1(x,\dot{x}) + g_2(x,\dot{x},u)$ and for any piecewise continuous control $u(x,\dot{x})$ a function $g(x,\dot{x},u(x,\dot{x}))$ be continuous and satisfy the Lipschitz condition with respect to x, $\dot{x}$ in any bounded domain of variables x, $\dot{x}$.

Let us re-write Eq. (3.129) accounting for discontinuity conditions. For a given function g we will have

$$\frac{dJ}{d\psi} = -\varepsilon\Big[rJ\delta_-^{2\pi}(\psi) + 4g_1\big(-J\chi, -J\omega\chi_\psi\big)\chi_\psi + 4g_2\big(-J\chi, -J\omega\chi_\psi, u\big)\chi_\psi\Big]$$
$$+\varepsilon^2\rho(J,\psi). \tag{3.166}$$

The rest term $\rho(J,\psi)$ is uniformly bounded in any domain of J and piecewise continuous with respect to ψ.

For assumptions introduced in [33], it follows that the partial averaging holds true. Let us confront (3.166) with a partially averaged system

$$\frac{d\tilde{J}}{d\psi} = -\frac{\varepsilon r}{2\pi}\tilde{J} - 4\varepsilon g_1^*\big(\tilde{J}\big) - 4\varepsilon g_2\big(-\tilde{J}\chi, -\tilde{J}\omega\chi_\psi, u\big)\chi_\psi \tag{3.167}$$

with the same boundary conditions, where

$$g_1^*(J) = \frac{1}{2\pi}\int_0^{2\pi} g_1\big(-J\chi(\psi), -J\omega(J)\chi_\psi(\psi)\big)\chi_\psi(\psi)d\psi.$$

Following conclusions of [33], we can obtain the estimate [comp.(3.118)]

$$\big|J(\tau) - \tilde{J}(\tau)\big| \le C\varepsilon. \tag{3.168}$$

Let us construct a control $u(\psi)$, $|u| \le U_0$, optimizing the system (3.167) with respect to the high-speed action. The Hamiltonian of the problem has the form

$$\tilde{H} = -\varepsilon\Big\{\omega^{-1}(J) + \tilde{p}\Big[\tilde{J}\big(r/2\pi + 4g_1^*\big(\tilde{J}\big)\big) + 4g_2\big(-\tilde{J}\chi, -\tilde{J}\omega\chi_\psi, u\big)\Big]\Big\},$$

a Lagrange multiplier satisfies the equation adjoint with (3.167) and not containing discontinuity conditions

$$\frac{d\tilde{p}}{d\psi} = \varepsilon\Big\{\frac{r}{2\pi}\tilde{p} + 4\tilde{p}\Big[g_1^*\big(\tilde{J}\big) + 4g_2\big(-\tilde{J}\chi, -\tilde{J}\omega\chi_\psi, u\big)\chi_\psi\Big]_J\Big\} + \big(\omega^{-1}\big(\tilde{J}\big)\big)_J, \tag{3.169}$$

and the optimal control can be found from the maximum condition

$$u_*(\psi) = \arg\max_{|u|\le U_0} \tilde{H}\big(\psi, \tilde{J}(\psi), \tilde{p}(\psi), u\big). \tag{3.170}$$

Substituting (3.170) into (3.169) and averaging, we will get equations identical to (3.136).

A possibility of a partial averaging in systems with functionals of the more general kind is proved in the same way as for smooth systems (Section A.2).

3.3.5
Main Motion Equations of the System with Double-Sided Constraints

Let the motion of an autonomous system be described by the equation

$$\ddot{x}+\Omega^2 x=\varepsilon g(x,\dot{x}) \tag{3.171}$$

and by conditions of an impact against a double-sided limiter

$$|x|=\Delta\,, \qquad \dot{x}_+=-(1-\varepsilon r)\dot{x}_-\,. \tag{3.172}$$

In order to determine a replacement of variables, that reduces the system to the standard form, consider a generating conservative system

$$\begin{aligned}&\ddot{x}+\Omega^2 x=0\,,\\ &|x|=\Delta\,, \qquad \dot{x}_+=-\dot{x}_-\,.\end{aligned} \tag{3.173}$$

A general integral of the system (3.173) has the form

$$x(t)=-J\chi_2(\psi,\omega)\,, \qquad \psi=\omega(J)(t-t_0)\,, \tag{3.174}$$

where $\chi_2(\psi,\omega)$ is a periodic Green's function of the second kind

$$\chi_2(\psi,\omega)=\frac{1}{\pi\omega}\sum_{k=-\infty}^{\infty}\frac{\exp[(2k-1)i\psi]}{(\Omega/\omega)^2-(2k-1)^2}\,, \tag{3.175}$$

or, in a closed form,

$$\chi_2(\psi,\omega)=\frac{1}{2\Omega}\frac{\sin\dfrac{\Omega}{\omega}(\psi-\pi/2)}{\cos(\pi\Omega/2\omega)}\,. \tag{3.176}$$

A dependence of the frequency on the impulse is given by the impact condition

$$x(0)=\Delta\,, \qquad J=-2\Omega\Delta\cot(\pi\Omega/2\omega)\,. \tag{3.177}$$

Introducing new variables 'impulse – phase' with a help of relations analogous to (3.108)

$$x=-J\chi_2(\psi,\omega)\,, \qquad \dot{x}=-\omega J\chi_{2\psi}(\psi,\omega)\,, \tag{3.178}$$

we will get after respective transformations equations of the type (3.114)

$$\begin{aligned}&\dot{J}=-4\varepsilon g(-J\chi_2,-J\omega\chi_{2\psi})\omega\chi_{2\psi}\,, \qquad \omega=\omega(J)\\ &\dot{\psi}=\omega\left[1+4\varepsilon J^{-1}g(-J\chi_2,-J\omega\chi_{2\psi})(J\chi_2)_J\right].\end{aligned} \tag{3.179}$$

Impact conditions are reduced to discontinuity conditions with respect to the variable J

$$J_{+}^{2k} - J_{-}^{2k} = -\varepsilon r J_{-}^{2k}, \qquad \psi = 2\pi k,$$
$$J_{+}^{2k-1} - J_{-}^{2k-1} = -\varepsilon r J_{+}^{2k-1}, \qquad \psi = \pi(2k-1). \tag{3.180}$$

Substituting the discontinuity conditions into the motion equations, we get

$$\dot{J} = -\varepsilon\omega\left[rJ\delta_{2-}^{2\pi} + 4g(-J\chi_2, -J\omega\chi_{2\psi})\chi_{2\psi}\right],$$
$$\dot{\psi} = \omega\left[1 + 4\varepsilon J^{-1} g(-J\chi_2, -J\omega\chi_{2\psi})(J\chi_2)_J\right], \tag{3.181}$$

where

$$J\delta_{2-}^{2\pi}(\psi) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(\psi - \pi k) J_{-}^{k};$$

impact moments t_k are determined by a condition

$$\psi(t_k) = \pi k. \tag{3.182}$$

A solution of the system (3.181) is approximated by the solution of the averaged system

$$\dot{J}^0 = -\varepsilon\left[2rJ^0 T_0^{-1} + 4\omega g_1'(J^0)\right],$$
$$\dot{\psi} = \omega\left[1 + 4\varepsilon g_2'(J^0)\right], \qquad \omega = \omega(J_0), \qquad T_0 = 2\pi/\omega(J_0), \tag{3.183}$$

where

$$g_1'(J) = \frac{1}{2\pi}\int_0^{2\pi} g(-J\chi_2(\psi), -J\omega\chi_{2\psi}(\psi))\chi_{2\psi}(\psi)d\psi,$$
$$g_2'(J) = \frac{1}{2\pi J}\int_0^{2\pi} g(-J\chi_2(\psi), -J\omega\chi_{2\psi}(\psi))(J\chi_{2\psi}(\psi))_J d\psi \tag{3.184}$$

and the estimate (3.118) holds.

4 Oscillations in Systems with Random Disturbances

Some problems of stochastic dynamics in oscillatory systems are examined in this Chapter.

Numerous papers and monographs deal with analysis of random oscillations and elaboration of various approximated methods for their investigations (see, for instance, [28, 37, 47, 120, 121, 129, 179]). Still, common methods of an analytical study of dynamic systems with random disturbances are not elaborated up to now. Exact results can be obtained only for the cases when the system dynamics is described by Itô's (or Stratonovitch') stochastic equations. Then it is possible to formulate Fokker-Planck-Kolmogorov equations for probability density of an output process and to solve them either analytically or numerically. If an excitation can be described as a component of a Markov process, then a space of 'sufficient coordinates' is introduced in such a way that an expanded system could be described by Itô's equations [121]. Disadvantages of such an approach are obvious: firstly, it needs an expansion of the space of phase coordinates that considerably complicates all numerical procedures, secondly, it unnecessary concretizes the properties of an input excitation, often known only from experimental data treatment.

More simple and approved in practice approaches are related to an assumption on the closeness of the output process to a normal one. If the system is linear and an external excitation is a normal process then the output process is also normal. If the system is non-linear then the assumption on the closeness of the output process to a normal one allows an utilization of some heuristic methods, such as a method of moment functions [28], a method of stochastic linearization [82, 109] or of a linearization with respect to a distribution function [82]. Still, the absence of the accuracy estimates for solutions complicates their application.

Numerical procedures analogous to the disturbance method seem to be more advantageous. Suggestions that disturbances are small or small 'in average' and rapidly change if compared to deterministic components can be more natural. Numerous works are concerned with an elaboration of mathematical procedures on the basis of disturbance methods; a detailed bibliography is given, for instance, in [28, 37, 179].

In this Chapter not all the possible schemes of small disturbances and methods of their investigations are given. Limit theorems (a stochastic analog of the averaging method – a method of diffusive approximation), allowing the singling out of a diffusive component from a disturbed movement and the construction of a generating operator of the system, prove to be an effective way for an asymptotic analysis. The main result is an approximate replacement of the disturbed system with a system of simpler structure, the methods of analysis for which are known.

All the material of Chapter is treated as additional, destined for further investigations of control problems. Therefore, the main attention is being paid to a formulation of conditions, under which the asymptotic approach holds true, and to the application of limit theorems to particular problems of the oscillation theory.

The necessary information about the theory of stochastic differential equations is given in Section 4.1.

The method of diffusive approximation in the form, in which it is used in this book, was suggested in the works of T.G. Kurtz [74] and H.J. Kushner [179]. The dynamics of non-oscillatory systems in weak force fields served as an application. In other words, coefficients in equations in the standard form did not depend on 'fast' variables, and a 'fast' time parameter was introduced only via an external excitation. An averaging procedure was reduced to an averaging with respect to realizations with account for a mixing properties of random processes. In oscillatory systems, coefficients of equations in the standard form depend directly on the 'fast' time parameter, and the averaging procedure involves also an averaging with respect to time.

The main results of the method of diffusive approximation and their generalization for systems with a fast rotating phase are given in Appendix, Section A.6.

The method of diffusive approximation and other asymptotic methods are broadly used for solution of applied problems of stochastic dynamics (see, for instance, [28, 37, 96, 107, 136, 179]). A monograph of R.L. Stratonovitch [121], where a limit behavior of dynamic systems under a wide-band excitation was for the first time thoroughly studied with a physical level of strictness, should be mentioned separately. In works concerned with the analysis of oscillatory systems, a general algorithm of an approximate solution is usually given, and a solution for systems with one or two degrees of freedom (moment characteristics, distribution functions, etc.) are written. In Section 4.2, averaged differential equations for a system with n degrees of freedom are directly formulated. Equations for the first and second moments for linear systems are obtained.

In Section 4.3, main results of a disturbed motion convergence to a homogeneous diffusion process for an infinite time interval are given.

In Section 4.4, a generalization for the method of diffusive approximation for a case of vibroimpact systems is given.

Averaging algorithms, formulated in Sections 4.2 – 4.4, are used in Chapter 5 for a construction of a quasi-optimal control in systems with a random excitation.

4.1 Stochastic Differential Equations

Some ideas of the theory of random processes and stochastic differential equations are given below. More detailed elucidation can be found, for instance, in [45, 46, 53, 129].

A main statistic characteristic of a vector random parameter $\xi \in R_l$ is its distribution function $F_1(x_1,...,x_l)$ equal to an event probability $(\xi_1 < x_1,...,\xi_l < x_l)$, i.e.,

$$F(x_1,...,x_l) = P(\xi_1 < x_1,...,\xi_l < x_l), \tag{4.1}$$

where ξ_i are components of the vector ξ.

Sometimes it is preferably to characterize random magnitudes not by the distribution function, but by the probability density $p(x_1,...,x_l)$ linked with F by relations

$$F(x_1,...,x_l) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_l} p(x_1,...,x_l)dx_1...dx_l,$$

or

$$p(x_1,...,x_l) = \frac{\partial^l F(x_1,...,x_l)}{\partial x_1 ... \partial x_l}. \tag{4.2}$$

Time t is treated in analysis of random processes $\xi(t)$ as a parameter having some set of values T. Owing to principles of the probability theory, a random process $\xi(t)$ is considered to be determined if all possible distribution functions of a finite dimension

$$F(x_1,...,x_l) = P\{\xi(t_1) < x_1,...,\xi(t_n) < x_n\}, \quad n = 1,\ 2,\ ...,$$

are given (if ξ is a vector process then inequalities should be fulfilled for each component).

Important characteristics of the random process are its expected value $m(t) = M\xi(t)$ and a correlation matrix

$$K(s,t) = M\left\{[\xi(s) - m(s)][\xi(t) - m(t)]'\right\}.$$

Here (an below) a prime as a superscript is used to denote a transpose.

An arbitrary random process $\xi(t)$ following conditions

$$m(t) = \text{const}, \qquad K(s,t) = K(s-t)$$

is called *stationary in a wide sense.* A Fourier transformation

$$S(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\tau) e^{i\lambda\tau} d\tau$$

of the matrix $K(\tau)$ is called *a matrix of spectral densities of the process* $\xi(t)$.

A full description of the random process $\xi(t)$ is possible only with infinite–dimensional distributions.

A direct construction of such distribution functions is practically unrealizable. Therefore, the role of different models of random processes based on distributions with a finite dimension is very important.

Disturbances acting on mechanical systems can be often presented in the form of normal Gauss random processes [28, 47, 118]. Even in the case when an excitation is a sum of independent harmonics, it can be approximated by a normal random process [82].

Let us mention the properties of the Gauss process.

A joint probability density for values $\xi^i=\xi(t_i)$ of a one–dimensional normal process $\xi(t)$ in any n moments t_i is given by relation

$$p(\xi^1,\dots,\xi^n)=\frac{1}{\sqrt{(2\pi)^n\Delta}}\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n}\frac{\Delta_{kj}(\xi^j-m^j)(\xi^k-m^k)}{\Delta}\right\}.$$

Here $m^j=M\xi(t_j)$, $\Delta=\det K$, $K=\{K_{jr}\}_{j,r=1}^{n}$ is a correlation matrix with elements

$$K_{jr}=M\left\{\left[\xi(t_j)-m^j\right]\left[\xi(t_r)-m^r\right]'\right\},$$

Δ_{rj} are minors for elements of the matrix K.

Exhaustive information on properties of the Gauss process is in its first two distribution moments of this process. The moments of the higher order are algebraic functions of these two moments and are expressed with the help of recurrent relations [53].

A process $\xi(t)$, $t\in T$, is called a *process with independent increments* if for any set t_1, ..., t_m, $0<t_1<\ldots<t_m<T$, random values $\xi(t_1)-\xi(0)$,..., $\xi(t_m)-\xi(t_{m-1})$ are mutually independent. The Gauss process with independent increments is called *a process of Brownian motion* (or *Wiener process*).

The Wiener process can serve as a model for a description of a motion of a microscopic particle in fluid. As was shown by N. Wiener, that a trajectory of the motion of the microscopic particle in such a model has no tangent, i.e., nearly all realizations of the Wiener process are continuous but nowhere differentiable functions [45, 53].

The Wiener process with characteristics

$$M[\xi(t)-\xi(s)]=0,\qquad M[\xi(t)-\xi(s)]^2=t-s \tag{4.3}$$

for all $s\le t$ is called a standard Wiener process and is denoted $w(t)$. It was proved that any continuous with probability 1 process with independent increments is a Wiener process.

The Wiener process generates an important class of random processes which are called *Markov processes*. In order to determine a Markov process we will introduce a notion of conditional probability.

Let $\xi(t)$ be a random process in R_l, A is some interval in R_l (we will understand under an interval $A = J[a,b)$ in R_l a set of points $\{x_1,...,x_n\}$, $a_i \le x_i < b_i$, $i = 1,...$; intervals $J(a,b)$, $J(a,b]$, $J[a,b]$ can be determined analogously). Let further an event α be determined by a condition $\xi(t) \in A$, i.e., a probability of such an event $P(\alpha) = P\{\xi(t) \in A\}$. Suppose that a probability for an event β is known: $P(\beta) = P\{\xi(u) \in A\}$. Then a conditional probability of any event α for the hypothesis β (under conditioned that the event β occurs) is

$$P(\alpha/\beta) = P\{\xi(t) \in A | \xi(u) \in A\} = P\{\xi(t) \in A | \xi(u)\}. \tag{4.4}$$

A process $\xi(t)$ is called *a Markov process* when the conditional probability determined in (4.4) is

$$P\{\xi(t) \in A | \xi(t_1), \xi(t_2), ..., \xi(t_n)\} = P\{\xi(t) \in A | \xi(t_n)\} \tag{4.5}$$

for any intervals $A \in R_l$, $t_1 < t_2 < ... < t_n < t$.

The Markov process can be non-formally determined as a process the value of which at a current moment t_0 entirely determines its future behavior for $t > t_0$ independent of its trajectory for $t < t_0$. A Wiener process can serve as an example of a Markov process. Another obvious example of the Markov process is a solution of an ordinary deterministic differential equation, although here a random factor can be involved only in initial conditions.

Let now a system motion depend on random factors and be described by an equation of the form

$$\dot{x} = b(t,x) + \sigma(t,x)\dot{\xi}(t), \quad x(t_0) = a, \tag{4.6}$$

where $\dot{\xi}(t)$ is some random process. Then characteristics of the process $x(t)$ for any interval $t_0 \le t \le T$ depend on initial conditions $x_0(t)$ and on statistical characteristics of the process $\dot{\xi}(t)$ for this interval. It is obvious, that the process $x(t)$ does not depend on the history for $t < t_0$ only in the case when the values of the excitation $\dot{\xi}(t)$ at any two moments $t_1 < t_0$ and $t_2 > t_0$ are statistically independent. It means that the process $\dot{\xi}(t)$ should be a continuous process with independent values. Though such physical process does not exist, still Eq. (4.6) can be given a strict sense. For this reason it is expedient to write this equation in the differential form:

$$dx = b(t,x)dt + \sigma(t,x)d\xi(t).$$

The process $\xi(t)$ should possess the following properties. Firstly, it should have independent increments as an integral of the 'process' $\dot{\xi}(t)$. Secondly, it should have continuous trajectories since only in this case the solution $x(t)$ will be a continuous random process. It was proved [45] that any process with such properties is a Wiener process. It can be also considered that its characteristics satisfy conditions (4.1), since it can be always reached changing coefficients b and σ.

Thus, a solution of the equation

$$dx = b(t,x)dt + \sigma(t,x)dw, \tag{4.7}$$

where $x \in R_n$, $w(t)$ is a l-dimensional standard Wiener process, $b(t,x)$, $\sigma(t,x)$ are matrices of respective dimensions, is a n–dimensional Markov process.

A mathematical theory of equations of the kind (1.7) (Itô's stochastic differential equations) was sufficiently detailed developed (see, for instance, [46, 129]). Let us give the main results, concerning the properties of solutions of Eq. (4.7), without proof.

Theorem 4.1. *Let functions $b(t,x)$ and $\sigma(t,x)$ be continuous with respect to t, x for $t \in [t_0,T]$, $x \in R_n$ and satisfy conditions*

$$\begin{gathered} |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C|x-y|, \\ |b(t,x)|^2 + |\sigma(t,x)|^2 \le C\left(1+|x|^2\right). \end{gathered} \tag{4.8}$$

Then for any random value $x(t_0)$ not depending on the process $w(t) - w(t_0)$ there exists a unique solution of Eq. (4.7) *which is a continuous Markov process.*

Here the uniqueness is understood in the following sense: if $x_1(t)$ and $x_2(t)$ are two continuous solutions of Eq. (4.7) then $P\{x_1(t) = x_2(t)\} = 1$ for all $t \in [t_0,T]$.

Below the following notation is used: $X^{s,x}(t)$ is a solution of Eq. (4.7) for initial conditions $X^{s,x}(s) = x$. In cases when it is unambiguous, we will write simply $x(t)$ or $x(s)$ understanding under such a notion solutions of Eq. (4.7) for some initial conditions.

It follows from Theorem 4.1 that for all $t \in [t_0,T]$ a solution $x(t)$ is determined and continuous. But sufficient conditions (4.8) are too limiting. In [129] more general conditions of existence and uniqueness for solutions of Eq. (4.7) are formulated in terms of Lyapunov's functions. Below it is always considered that a solution of the equation under study exists and is unique.

Theorem 4.2 (Itô's differentiation formula). *If a function $V(t,x)$ has continuous partial derivatives up to the second order with respect to x and up to the first order with respect to t, and a process $\zeta(t)$ with values form R_l has a stochastic Itô's differential*

$$d\zeta = b(t)dt + \sigma(t)dw,$$

then a process $\eta(t) = V(t,\zeta(t))$ has also a stochastic Itô's differential, and

$$d\eta = \left[V_t(t,\zeta) + V_x'(t,\zeta)b(t) + \frac{1}{2}\operatorname{Tr} A(t)V_{xx}(t,\zeta)\right]dt + V_x'(t,\zeta)\sigma(t)dw,$$
$$A(t) = \sigma(t)\sigma'(t). \tag{4.9}$$

Here $\operatorname{Tr} A$ is a trace of a matrix A, a vector V_x has components $\partial V/\partial x_i$, a square matrix V_{xx} has components $\partial^2 V/\partial x_i \partial x_i$.

We will write a differentiation formula of a complex function on trajectories of Eq. (4.7).

Let a function $V(s,x)$ be two times continuously differentiable with respect to x and one time differentiable with respect to t. Then from Theorems 4.1 and 4.2 it follows, that

$$V(t,x(t)) - V(s,x(s)) = \int_s^t \mathcal{L}_u(u,x(u))du + \int_s^t V_x'(u,x(u))\sigma(u,x(u))dw(u), \tag{4.10}$$

where an operator

$$\mathcal{L}_s V = \frac{\partial V}{\partial s} + \mathcal{L}V, \quad \mathcal{L}V = b'(s,x)V_x + \frac{1}{2}\operatorname{Tr} A(s,x)V_{xx}. \tag{4.11}$$

Calculating owing to (4.10) an expected value, we get

$$M\left[V(t,x(t)) - V(s,x(s))\right] = \int_s^t M\mathcal{L}_u V(u,x(u))du. \tag{4.12}$$

Substituting this in the equality $x(t) = X^{s,x}(t)$, dividing both its parts with $t-s$ and taking a limit for $t \to s+0$, we will obtain

$$\lim_{h\to 0}\frac{1}{h}\left[M_{s,x}V(s+h,x(s+h)) - V(s,x)\right] = \mathcal{L}_s V(s,x). \tag{4.13}$$

Here $M_{s,x}$ is a conditional expected value

$$M_{s,x}V(t,x(t)) = MV(t,X^{s,x}(t)) = MV(t,x(t)/s,x).$$

An operator $\mathcal{L}_s$ determined by Eq. (4.13) is called a generating differentiation operator of the process $X^{s,x}(t)$. For Markov processes it can be expressed in the direct form (4.11).

If a respective Markov process is homogeneous ($b = b(x), \sigma = \sigma(x)$), and the function $V = V(x)$, then $\mathcal{L}_s V = \mathcal{L}V$.

Control problems are based on the study of mathematical properties of expected values for different functionals of solutions to stochastic differential equations. In some cases, a calculation of these values is reduced to a solution of the boundary-value problems for equations in partial derivatives. Suppose that we are to calculate, for instance, an expected value

$$V(s,x) = MF(X^{s,x}(t)), \tag{4.14}$$

where $F(x)$ is a given function, $X^{s,x}(t)$ is a solution of the stochastic equation (4.7).

Theorem 4.3 [129]. *Let*

1) $V(s,x)$ be a bounded and continuous for all $s \le t$, $x \in R_n$ function, satisfying in this domain the equation

$$\mathcal{L}_s V(s,x) = 0, \quad s \le t, \tag{4.15}$$

with a boundary condition

$$V(t,x) = F(x); \tag{4.16}$$

2) all conditions providing existence and uniqueness of the solution of Eq. (4.7) *be fulfilled for all $t \ge t_0$* [129].

Then Eq. (4.14) *holds true.* Here $\mathcal{L}_s$ is a generating operator of the process $X^{s,x}(t)$ which is determined by Eq. (4.11).

A solution of the Cauchy problem (4.15), (4.16) determines a value of the functional for arbitrary initial conditions (s,x).

In a general case, there are no theorems of existence and uniqueness for solutions of the problem (4.15), (4.16). It was proved [45, 129] that a solution $V(s,x)$ exists and coincides with values of the functional (4.14) when coefficients of Eq. (4.7) satisfy the conditions (4.8) and a condition of non-degeneracy

$$(A(t,x)\lambda,\lambda) \ge m(t,x)|\lambda|^2, \quad A = \sigma\sigma' \tag{4.17}$$

for $x \in R_n$, $t \in [0,T]$. Here $m(t,x)$ is a positive continuous function, λ is an arbitrary constant vector. The solution $X^{s,x}(t)$ of Eq. (4.7) is regular under the conditions (4.8), i.e., it does not tend to infinity in a finite time, and all the mentioned operations for expected values hold.

Suppose that Eq. (4.7) and the functional (4.14) satisfy these conditions. Let, further, a jointly unique non-linear replacement of variables

$$y = \varphi(x), \quad x = \psi(y) \tag{4.18}$$

be introduced, where φ and ψ are sufficiently smooth functions. It is obvious that then

$$MF\left(X^{s,x}(t)\right) = MF\left[\psi\left(Y^{s,y}(t)\right)\right] = MG\left(Y^{s,y}(t)\right) \tag{4.19}$$

(if all the operations are determined). Here $Y^{s,y}(t)$ is a solution of a stochastic differential equation obtained form Eq. (4.7) by means of the Itô's transformation, $y = Y^{s,y}(t)$. At the same time,

$$V(s,x) = V\left(s,\psi(y)\right) = W(s,y), \tag{4.20}$$

i.e., a function $W(s,y)$ can be found as a solution of Eq. (4.15), transformed by means of (4.18).

Just such cases are studied below. A non-linear replacement of variables transforms the coefficients of an initial equation in such a way that conditions of a linear growth are broken. But the process $Y^{s,y}(t)$ remains regular, and the function $W(s,y)$ is uniquely determined as a unique solution of the inverse Kolmogorov's equation. A correctness of the transitions (4.19), (4.20) for each particular case is not discussed.

4.2 Limit Theorems for Stochastic Differential Equations (The Diffusion Approximation Method)

Eq. (4.7) is the best examined model of a stochastic system. For such a system it is possible to obtain probability characteristics and to formulate Eqs. (4.15) – (4.17) which determine a character of various functionals. At the same time, it is easy to understand that Eq. (4.7) is an idealization of some physical system (4.6), an excitation $\xi(t)$ in which is close in a certain sense to a process with independent increments $w(t)$. Non-differentiability of the process $w(t)$ was often considered to be a disadvantage of the model, since in a course of a formulation of a mathematical model such an interpretation of a random excitation should be used, which should lead to an adequate description of a real system.

A problem of correct physical interpretation of stochastic differential equations was thoroughly studied by R. L. Stratonovitch [120, 121] and R. Z. Khasminsky [129].

Let us once more consider Eq. (4.6). Let $\dot{\xi}(t)$ be some δ-correlated stationary Gauss process. Obviously, owing to this δ-correlation, all finite-dimensional characteristics of the process $\dot{\xi}(t)$ for $t \geq t_0$ do not depend on its values for $t < t_0$. Accordingly, all the characteristics of the process $x(t)$ for $t \geq t_0$ also do not depend on the values $x(t)$ for $t < t_0$. Hence, the δ-correlated Gauss excitation („white noise") also generates in the system (4.6) a Markov process.

R. L. Stratonovitch has shown [120, 121] that a system of equations

$$\dot{x}_i = b_i(t,x) + \sum_{k=1}^{l} \sigma_{ik}(t,x)\dot{\xi}_k^*(t), \tag{4.21}$$

where $\dot{\xi}_k^*(t)$ is continuous on the right δ-correlated process of the „white noise" type, is equivalent to the following system of Itô's stochastic equations

$$dx_i = \left[b_i(t,x) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{l} \frac{\partial \sigma_{ik}(t,x)}{\partial x_j}\sigma_{jk}(t,x) \right] + \sum_{k=1}^{l} \sigma_{ik}(t,x)dw_k\,, \tag{4.22}$$

where w_k are one-dimensional standard independent Wiener processes.

Thus, a generating differentiation operator of the Markov process, determined by Eq. (4.21), has the form

$$\mathcal{L}_s = \frac{\partial}{\partial s} + \mathcal{L}^*, \tag{4.23}$$

$$\mathcal{L}^* = \sum_{i=1}^{n}\left[b_i(t,x) + \frac{1}{2}\sum_{k=1}^{l}\sum_{j=1}^{n} \frac{\partial \sigma_{ik}(t,x)}{\partial x_j}\sigma_{jk}(t,x) \right]\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t,x),$$

where still

$$a_{ij} = \sum_{k=1}^{l}\sum_{j=1}^{n} \sigma_{ik}\sigma_{jk}\,.$$

The operator (4.23) has – as distinct from (4.11) – additional terms

$$\frac{1}{2}\sum_{k=1}^{l}\sum_{j=1}^{n} \frac{\partial \sigma_{ik}(t,x)}{\partial x_j}\sigma_{kj}$$

in a drift coefficient.

Thus, Eqs. (4.7) and (4.21) correspond to different physical models. In physical problems where the „white noise" represents an idealization of the real process with a small correlation time, the respective equation should be understood as the Stratonovitch stochastic equation with the generating differential operator (4.23).

A direct replacement of Eq. (4.21) with Eq. (4.7) leads to false results (these questions are treated in detail in [129]).

Let, for instance, a motion equation of a mechanical system be reducible to the form

$$\dot{x} = \varepsilon f(x)\xi(t) + \varepsilon^2 g(x), \tag{4.24}$$

where ε is a small parameter, $x \in R_n$, $\xi(t)$ is a scalar stationary random process with a correlation function $K(z)$ and a spectral density $S(\omega)$, $M\xi(t) = 0$. Let us show that an introduction of the small parameter characterizes a closeness of the excitation to the „white noise". Re-write (4.24) in the form

$$dx_\varepsilon / d\tau = f(x)\xi_\varepsilon(\tau) + g(x), \quad \tau = \varepsilon^2 t, \tag{4.25}$$
$$\xi_\varepsilon(\tau) = \varepsilon^{-1}\xi(\tau/\varepsilon^2).$$

Obviously, the process $\xi_\varepsilon(\tau)$ has the correlation function $K_\varepsilon(z) = \varepsilon^{-2}K(z/\varepsilon^2)$ and the spectral density $S_\varepsilon(\omega) = S(\varepsilon^2\omega)$. Thus, if the function $S(\omega)$ is continuous for $\omega \to 0$ then $S_\varepsilon(\omega) \to S_0 = S(0)$ for $\varepsilon \to 0$ and $K_\varepsilon(z) \to 2\pi S_0 \delta(z)$, respectively. In other words, for $\varepsilon \to 0$ the process $x_\varepsilon(\tau)$ converges to some Markov process $x_0(\tau)$ – the solution of the Stratonovitch equation.

Such a limit transition was for the first time studied by R. L. Stratonovitch [120]. A strict proof was given by R. Z. Khasminsky [132]. In posterior works, in particular [30, 188, 190, 196], conditions imposed on the coefficients of the equations and random excitations were specified.

Below the asymptotic *method of diffusion approximation*, given in detail in [179], is used for construction of the approximated solution.

Before the formulation of Theorems about the limit transition in stochastic differential equations, let us recall definitions of various types of convergence [27].

Let x and x_n be random variables in R_l, $F(\xi)$ and $F_n(\xi)$ be distribution functions, generated by these values. A succession of distribution functions is known to converge weakly to F for $n \to \infty$ when

$$F_n(\xi) \to F(\xi), \quad n \to \infty, \tag{4.26}$$

for each value of ξ which is a continuity point of the limit function $F(\xi)$ (including the points $\xi = \pm\infty$). A weak convergence in the sense of (4.26) is equivalent to the convergence

$$Mf(\xi_n) \to Mf(\xi), \tag{4.27}$$

for $n \to \infty$ for any continuous and bounded function $f(\xi)$. If (2.6) holds then the succession x_n *converges weakly* to x for $n \to \infty$.

A succession x_n is called *weakly compact* if it is possible to single out from it a weakly converging sub-succession. For a weak compactness of the succession x_n it is sufficient that values x_n are uniformly bounded with respect to probability, i.e.,

$$\sup_n P\{|x_n| > R\} \to 0, \quad R \to \infty. \tag{4.28}$$

For a succession of random processes $x_n(t)$, $t \in [t_0, T]$, the condition (4.28) gets the form

$$\sup_{n,t} P\{|x_n| > R\} \to 0, \quad R \to \infty. \tag{4.29}$$

Let the processes $x_n(t)$, $x(t)$ be continuous on the segment $[t_0, T]$. Let $C[t_0, T]$ denote a space of continuous on $[t_0, T]$ functions to which belong (with probability 1) all the trajectories of the processes $x_n(t)$, $x(t)$. The succession $x_n(t)$ is called *weakly converging* to $x(t)$ if for any continuous on $C[t_0, T]$ functional

$$Mf(x_n(t)) \to Mf(x(t)). \tag{4.30}$$

The weak convergence of the succession $x_n(t)$ in a space $D[t_0, T]$ without discontinuities of the second kind is analogously determined (under respective introduction of a norm [179]).

The limit theorems for stochastic differential equations determine conditions of the weak convergence of the solution succession $x_n(t)$ to a solution $x(t)$ of some limit equation.

The following statement holds [27]: *if finite–dimensional distributions of the process $x_n(t)$ converge weakly to finite–dimensional distributions of the process $x(t)$ and the succession $x_n(t)$ is weakly compact then the condition* (4.30) *holds.*

Thus, two points are stated for a proof of the condition (4.30): a convergence of finite–dimensional distributions of the succession $x_n(t)$ to some limit distributions and a weak compactness of the succession $x_n(t)$.

The proof of these two points makes a content of the limit theorem for stochastic differential equations.

Principles of construction of an approximate solution were given in [174, 179]; a systematic account of the asymptotic method for systems of the type (4.24), (4.25) is given in the monograph [179]. In Section A.6, main theorems of the method of diffusion approximation and their generalization to the case of stochastic systems with a fast rotating phase are formulated.

Through the rest of the Section, we will limit the examination to systems, motion equations of which can be reduced to the standard form

$$\dot{x} = \varepsilon F(t, \tau, x, \xi(t)) + \varepsilon^2 G(t, \tau, x), \quad x(0) = a. \tag{4.31}$$

Here $x \in R_n$, $\xi \in R_l$, $\tau = \varepsilon^2 t$ is a small parameter. It is considered that functions F, G and random excitations $\xi(t)$ satisfy conditions (A) and (B) of Theorem A.14, and the system (4.31) can be studied by means of the method of diffusive approximation.

Let us give without the proof main results.

Introduce the following notation for mixed moments

$$K(s,t,\tau,x) = M\left[F_x(s,\tau,x,\xi(s))F(t,\tau,x,\xi(t))\right],$$
$$A(s,t,\tau,x) = M\left[F'(s,\tau,x,\xi(s))F(t,\tau,x,\xi(t))\right], \tag{4.32}$$

i.e., elements k_j of a vector k and a_{ij} of the matrix A have the form

$$k_j = M\sum_{r=1}^{n} \frac{\partial F_j(s,\tau,x,\xi(s))}{\partial x_r} F_r(t,\tau,x,\xi(t)),$$
$$a_{ij} = M\left[F_i(s,\tau,x,\xi(s))F_j(t,\tau,x,\xi(t))\right]. \tag{4.33}$$

Let, further, limits

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} G(t,\tau,x)dt = \overline{G}(\tau,x),$$
$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} ds \int_{t_0}^{T+t_0} A(s,t,\tau,x)dt = A(\tau,x), \tag{4.34}$$
$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} ds \int_{s}^{T+t_0} K(s,t,\tau,x)dt = K(\tau,x)$$

exist, continuously with respect to $\tau \in [0,\tau_f]$, $x \in K \subset R_n$, $t_0 \ge 0$. Let $x_\varepsilon(t)$ denote a solution of the equation

$$dx_\varepsilon/d\tau = \varepsilon^{-1}F\left(\tau/\varepsilon^2,\tau,x,\xi(\tau/\varepsilon^2)\right) + G\left(\tau/\varepsilon^2,\tau,x_\varepsilon\right), \tag{4.31a}$$
$$x_\varepsilon(0) = a.$$

If coefficients of the system (4.31) satisfy conditions of Theorem A.14 then for $0 < \varepsilon \le \varepsilon_0$, $0 \le \tau \le \tau_f$ the process $x_\varepsilon(t)$ converges weakly to a continuous with probability 1 Markov process $x_0(t)$ – a solution of the equation

$$dx_0(\tau) = b(\tau,x_0)d\tau + \sigma(\tau,x_0)dw, \qquad x_0(0) = a \tag{4.35}$$

(if such a solution exists). Here

$$b(\tau,x)=K(\tau,x)+\overline{G}(\tau,x), \quad \sigma(\tau,x)\sigma'(\tau,x)=A(\tau,x). \tag{4.36}$$

Consider that components F_{ij} of the matrix F_0 and G_i of the vector G are periodic with respect to t with a period θ (i, $j=1$, ..., n), $\xi(t)$ is a n–dimensional stationary random process with a zero mean value and a correlation function $K(u)$. $K_{rm}(u)=M[\xi_r(t+u)\xi_m(t)]$ are components of this matrix. Then coefficients b_i, a_{ij} are calculated with the help of relations

$$G_i(\tau,x)=\frac{1}{\theta}\int_0^\theta G_i(t,\tau,x)dt,$$

$$K_i(\tau,x)=\sum_{j,r,m=1}^{n}\frac{1}{\theta}\int_0^\theta ds\int_{-\infty}^{0}\frac{\partial F_{ir}(s,\tau,x)}{\partial x_j}F_{jm}(s+u,\tau,x)K_{rm}(u)du, \tag{4.37}$$

$$a_{ij}(\tau,x)=\sum_{r,m=1}^{n}\frac{1}{\theta}\int_0^\theta ds\int_{-\infty}^{\infty}F_{ir}(s,\tau,x)F_{jm}(t,\tau,x)K_{rm}(t-s)dt.$$

Theorem A.14 essentially determines the first approximation to the diffusive process. The main algorithm of the construction of approximations of higher orders was for the first time given in [120, 121] and was connected with calculations of the consequent terms in the generating operator of the process $x_\varepsilon(t)$.

Another methods of construction of approximations of the higher order are suggested in [28, 32, 54, 61, 126].

Theorem A.14 and the based on it asymptotic method of analysis of stochastic differential equations (the method of diffusion approximation) plays the same part as the averaging method for deterministic systems. The main result is an approximate replacement of the disturbed system with a system of simpler structure, methods of analysis of which are known.

Really, Theorems of Section A.6 indicate the calculation method for functionals determined on trajectories of the limit diffusion process x_0. At the same time, a weak convergence of x_ε to x_0 means that any continuous functional $\Phi(x_\varepsilon)$, determined on the trajectories of the disturbed system, is approximated by the value $\Phi(x_0)$ of the same functional on the trajectories of the limit system. Thus, the diffusion approximation method makes an approximate calculation of functionals on the trajectories of the disturbed system possible [see (A.118)].

Asymptotic methods are widely used for solution of applied problems of stochastic dynamics [28, 47, 96, 107, 120, 179]. The monograph of R. L. Stratonovitch [120] should be mentioned separately, since the limit behavior of dynamic systems under the wide-band excitation was studied there for the first time with the physical level of strictness.

Such a way can be also effective for an approximate solution of optimal control problems, since the quasi-optimality of the control is estimated just according to the closeness of functionals (a weak convergence) and not according to the closeness of trajectories (a strong convergence). This question is studied in detail in Chapter 5.

Oscillations in systems with random excitations. Let us construct limit diffusion equations and determine moment characteristics of some disturbed oscillatory systems in order to illustrate the diffusion approximation method.

1) Let the motion equation of a system with n degrees of freedom be reduced to the form

$$\ddot{x} + \left[A + \varepsilon \Xi(t)\right]x + \varepsilon^2 H(x.\dot{x}) = \varepsilon Z(t),$$
$$x(t_0) = a_1, \quad \dot{x}(t_0) = a_2. \tag{4.38}$$

Here $x \in R_n$, $A = \operatorname{diag}\left\{\lambda_j^2\right\}_{j=1}^{n}$, H is a vector of additional non-linear and non-conservative forces, components $\xi_{kj}(t)$ of a matrix $\Xi(t)$ and $\zeta_j(t)$ of a vector $Z(t)$ are stationary and stationary connected processes with a zero mean and sufficiently fast diminishing correlation functions. It is supposed that eigenfrequencies of the system are not connected by resonant relations, i.e., $\lambda_j/\lambda_k \neq m/r$, $k \neq j$ (j, $k = 1, \ldots, n$; m, $r = 1, 2, \ldots$). By means of the replacement

$$x_j = e^{y_j}\cos\left(\lambda_j t + \varphi_j\right), \quad \dot{x}_j = -\lambda_j e^{y_j}\sin\left(\lambda_j t + \varphi_j\right) \tag{4.39}$$

the system (4.38) is reduced to the standard form

$$\dot{y}_\varepsilon = \varepsilon F^1\left(t, y_\varepsilon, \varphi_\varepsilon, \xi(t)\right) + \varepsilon^2 G^1\left(t, y_\varepsilon, \varphi_\varepsilon\right),$$
$$\dot{\varphi}_\varepsilon = \varepsilon F^2\left(t, y_\varepsilon, \varphi_\varepsilon, \xi(t)\right) + \varepsilon^2 G^2\left(t, y_\varepsilon, \varphi_\varepsilon\right), \tag{4.40}$$

where

$$y_\varepsilon = \left(y_1, \ldots, y_n\right), \quad \varphi_\varepsilon = \left(\varphi_1, \ldots, \varphi_n\right),$$
$$F_j^1 = \left(\lambda_j e^{y_j}\right)^{-1}\left[\sum_{i=1}^{n} \xi_{ji}(t) e^{y_i}\cos\left(\lambda_i t + \varphi_i\right) - \zeta_j(t)\right]\sin\left(\lambda_j t + \varphi_j\right),$$
$$F_j^2 = \left(\lambda_j e^{y_j}\right)^{-1}\left[\sum_{i=1}^{n} \xi_{ji}(t) e^{y_i}\cos\left(\lambda_i t + \varphi_i\right) - \zeta_j(t)\right]\cos\left(\lambda_j t + \varphi_j\right) \tag{4.41}$$
$$G_j^1 = \left(\lambda_j e^{y_j}\right)^{-1} h_j\left(t, y_\varepsilon, \varphi_\varepsilon\right)\sin\left(\lambda_j t + \varphi_j\right),$$
$$G_j^2 = \left(\lambda_j e^{y_j}\right)^{-1} h_j\left(t, y_\varepsilon, \varphi_\varepsilon\right)\cos\left(\lambda_j t + \varphi_j\right),$$
$$h_j(t, y, \varphi) = H_j(x, \dot{x}).$$

A traditional replacement $x_j = R_j\cos\left(\lambda_j t + \varphi_j\right)$, $\dot{x}_j = -\lambda_j R_j\sin\left(\lambda_j t + \varphi_j\right)$

leads to an appearance of proportional to R_j^{-1} terms in equations in the standard form. In the presence of features of this kind, conditions of Theorem A.14 break. Therefore, variables y, φ are introduced according to Eq. (4.39) instead of the „amplitude – phase“ variables. In such a case, continuity conditions mentioned in the condition (B) of Theorem hold.

It can be easily shown that drift and diffusion coefficients of the limit stochastic equation calculated with the help of Eqs. (4.32) – (4.34) do not depend on φ, i.e.,

$$dy_0 = b^1(y_0)d\tau + \sigma^{11}(y_0)dw^1 + \sigma^{12}(y_0)dw^2,$$
$$d\varphi_0 = b^2(y_0)d\tau + \sigma^{21}(y_0)dw^1 + \sigma^{22}(y_0)dw^2, \tag{4.42}$$

$$b_j^i(y) = \overline{G}_j^i(y) + K_j^i(y), \quad i = 1,2; \quad j = 1,\dots,n, \tag{4.43}$$

$$\overline{G}_j^i(y) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} G_j(t,y,\varphi)dt, \tag{4.44}$$

$$K_j^i(y) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} ds\int_s^T K_j^i(s,t,y,\varphi)dt,$$
$$K_j^i(s,t,y,\varphi) = M\sum_{r=1}^{n}\left\{\frac{\partial F_j^i(s,y,\varphi,\xi(s))}{\partial y_r}F_r^1(t,y,\varphi,\xi(t)) + \frac{\partial F_j^i(s,y,\varphi,\xi(s))}{\partial \varphi_r}F_r^2(t,y,\varphi,\xi(t))\right\}. \tag{4.45}$$

Matrices $\sigma^{ij}(y)$, $i,\ j = 1,2$ are determined from relations

$$\left[\sigma^{ij}(y)(\sigma^{ij}(y))'\right] = A^{ij}(y), \tag{4.46}$$

$$A^{ij}(y) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} ds\int_{t_0}^{T+t_0} A^{ij}(s,t,y,\varphi)dt,$$
$$A_{km}^{ij}(s,t,y,\varphi) = M\left[F_k^i(s,y,\varphi,\xi(s))F_m^j(t,y,\varphi,\xi(t))\right]. \tag{4.47}$$

Substituting (4.41) into (4.47) and averaging, we will get $\overline{A}^{12} = \overline{A}^{21} = 0$. Thus, the problem is reduced to analysis of the limit stochastic differential equation

$$dy^0 = b^1(y^0)d\tau + \sigma^{11}(y^0)dw, \quad y^0(0) = \ln a, \tag{4.48}$$

where $w(\tau)$ is a n–dimensional standard Wiener process. Calculating components of the vector b^1 and the matrix $\overline{A}^{11} = \{a_{ij}\}_{i,j=1}^{n}$, we get

$$
\begin{aligned}
&b_j^1(y) = \nu_j + \overline{G}_j e^{-y_j}, \qquad a_{ij}(y) = \alpha_{ij}, \\
&a_{jj}(y) = \alpha_{jj} + d_{jj} e^{-2y_j} + q_j e^{-2y_j},
\end{aligned}
\tag{4.49}
$$

where the functions $\overline{G}_j(y)$ are calculated according to the relations (4.44) and

$$
\begin{aligned}
&\nu_j = \sum_{k=1}^{n} \alpha_{jk} \\
&q_j = \sum_{\substack{k=1 \\ k \neq j}}^{n}{}' q_{jk} e^{2y_k}, \\
&q_{jk} = \frac{1}{8\lambda_j^2}\left[S_{jk.jk}^{\xi}(\lambda_j + \lambda_k) + S_{jk.jk}^{\xi}(\lambda_j - \lambda_k)\right], \\
&\alpha_{jk} = \frac{1}{8\lambda_j \lambda_k}\left[S_{jk,kj}^{\xi}(\lambda_j + \lambda_k) - S_{jk,kj}^{\xi}(\lambda_k - \lambda_j)\right], \qquad j \neq k, \\
&\alpha_{jj} = \frac{1}{8\lambda_j^2} S_{jj}^{\xi}(2\lambda_j), \qquad d_{jj} = \frac{1}{2\lambda_j^2} S^{\zeta}(\lambda_j).
\end{aligned}
\tag{4.50}
$$

Here

$$
\begin{aligned}
&S_{jk,mr}^{\xi}(\omega) = \int_{-\infty}^{\infty} K_{jk,mr}^{\xi}(t) \cos \omega t dt, \\
&K_{jk,mr}^{\xi}(t) = M\left[\xi_{jk}(s+t)\xi_{mn}(s)\right], \\
&S_{jj}^{\xi} = S_{jj,jj}^{\xi} \\
&S_{jj}^{\zeta} = \int_{-\infty}^{\infty} K_{jj}^{\zeta}(t) \cos \omega t dt, \\
&K_{jj}^{\zeta}(t) = M\left[\zeta_j(s+t)\zeta_j(s)\right],
\end{aligned}
\tag{4.51}
$$

i.e., $S^{\xi}(\omega)$ and $S^{\zeta}(\omega)$ are joint spectral densities of respective excitations.

Knowing the drift and diffusion coefficients, it is possible to construct the Fokker-Planck-Kolmogorov equation and to determine a limit probability density and moment characteristics of an oscillation amplitude. In some particular case, an analytic solution can be found.

Let us calculate, for instance, second moments

$$
D_j = M\left[x_j^2 + \lambda_j^{-2} \dot{x}_j^2\right]
$$

of solutions of a quasi-linear system, for which $H(x, \dot{x}) = 2\beta\dot{x} + G(x)$, where β is a positively determined matrix of dissipation coefficients with components β_{ij}, i, $j = 1, \dots, n$. In this case $x_j^2 + \lambda_j^{-2}\dot{x}_j^2 = R_j^2$ and $\overline{G}(R) = -\beta_{jj}R$, $\beta_{jj} > 0$.

If $R_j(t,\varepsilon) = \exp y_j$, where y_j is a solution of the system (4.40), and $R_j^0(\tau,\varepsilon) = \exp y_j^0$, where y_j^0 is a solution of Eq. (4.48), $\tau = \varepsilon^2 t$, then

$$\left| MR_j^2 - M\left(R_j^0\right)^2 \right| \le C\varepsilon .$$

Applying the Itô's formula (4.9), we will write the system of equations for variables $z_j = \left(R_j^0\right)^2 = \exp\left(2y_j^0\right)$

$$dz_j = 2\left[\left(v_j - \beta_{jj} + \alpha_{jj}\right)z_j + \sum_{\substack{k=1 \\ k\neq j}}^{n}{}' q_{jk} z_k + d_{jj}\right]d\tau + 2z_j\sum_{k=1}^{n}\sigma_{jk}dw_k . \tag{4.52}$$

Coefficients σ_{jk} are determined by the relation $\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} = \alpha_{jk}$.

Calculating expected values of both part of Eq. (4.52), we get

$$\frac{dM_z}{d\tau} = QM_z + 2\delta , \tag{4.53}$$

where $M_z = \left(Mz_1, \dots, Mz_n\right)$, $\delta = \left(d_{11}, \dots, d_{nn}\right)$, Q is a matrix with elements

$$Q_{jk} = q_{jk}, \quad j \ne k, \quad Q_{jj} = 2\left(v_j - \beta_{jj} + \alpha_{jj}\right). \tag{4.54}$$

For a mean-square asymptotic stability of the initial system (4.38) it is necessary that all eigenvalues of the matrix Q are in a left-hand half-plane. In particular, for a system with one degree of freedom, Eq. (4.53) gets the form

$$\frac{dM_z}{dz} = Q_{11}M_z + 2d_{11}, \tag{4.55}$$

where $Q_{11} = -2\beta + S^{\xi}(2\lambda)/2\lambda^2$, $d_{11} = S^{\xi}(\lambda)/2\lambda^2$, and the stability condition $Q_{11} < 0$ leads to the well-known relation

$$S^{\xi}(2\lambda) < 4\lambda^2\beta . \tag{4.56}$$

Relations (4.50), (4.53) reveal the known effect – a stochastic analog of parametric resonances. It is clear that elements of the matrix Q depend only on spectral densities of parametric disturbances only for frequencies of main parametric resonances, and stability conditions of the solution are also determined by these relations. If the spectral density for one of the frequencies of the parametric resonance is high then the system destabilizes.

A further detailing makes possible a determination of the analogs of higher parametric resonances, but they can be found only with the use of higher approximations [28, 61, 101].

External disturbances $\xi_j(t)$ do not influence stability conditions, but a resonant effect takes also place, since elements δ_j of the vector δ depend on the spectral densities $S_{jj}^{\xi}(\lambda_j)$ for corresponding eigenfrequencies of the system.

In [47 – 49] an analysis of systems with external and parametric excitations is carried out for problems of identification and diagnostics.

Let us construct a stochastic Cauchy matrix $K_\varepsilon(t, t_0)$ of a disturbed system. In other words, we will find a solution of the linear system

$$\ddot{x} + \left[A + \varepsilon \Xi(t)\right]x + 2\varepsilon^2 \beta \dot{x} = 0, \tag{4.57}$$

satisfying initial conditions

$$x(t_0) = 0, \quad \dot{x}(t_0) = I. \tag{4.58}$$

Here a matrix x has a dimension $n \times n$, I is a unit matrix, A, β, Ξ are the same as in the previous case (for simplicity it can be considered that $\beta = \operatorname{diag}\{\beta_{jj}\}_{j=1}^{n}$).

An utilization of the replacement (4.39) is inconvenient here, since the limit equations (4.42) are sufficiently non-linear. Let us introduce a replacement of variables owing to relations

$$\begin{aligned} x &= F_c(t)x^1 + F_s(t)x^2 \\ \dot{x} &= -\Lambda F_s(t)x^1 + \Lambda F_c(t)x^2, \end{aligned} \tag{4.59}$$

where

$$F_c = \operatorname{diag}\{\cos \lambda_j t\}, \quad F_s = \operatorname{diag}\{\sin \lambda_j t\},$$

$$\Lambda = \operatorname{diag}\{\lambda_j\}, \quad A = \Lambda^2, \quad j = 1, \ldots, n.$$

Let us reduce (4.57) to the standard form, keeping linearity with respect to new variables

$$\begin{aligned} \dot{x}^1 &= \varepsilon \Lambda^{-1} F_s(t) \Xi(t)\left(F_c(t)x^1 + F_s(t)x^2\right) + 2\varepsilon^2 F_s(t)\beta\left(-F_s(t)x^1 + F_c(t)x^2\right), \\ \dot{x}^2 &= \varepsilon \Lambda^{-1} F_c(t) \Xi(t)\left(F_c(t)x^1 + F_s(t)x^2\right) - 2\varepsilon^2 F_c(t)\beta\left(-F_s(t)x^1 + F_c(t)x^2\right), \end{aligned} \tag{4.60}$$

$$x^1(\tau_0) = -\Lambda^{-1} F_s(t_0), \quad x^2(\tau_0) = \Lambda^{-1} F_c(t_0),$$

or, in a more short form,

$$\dot{x}^1 = \varepsilon\left[\mu^{11}(t) + \varepsilon \nu^{11}(t)\right]x^1 + \varepsilon\left[\mu^{12}(t) + \varepsilon \nu^{12}(t)\right]x^2,$$

$$\dot{x}^2 = \varepsilon\left[\mu^{21}(t) + \varepsilon\nu^{21}(t)\right]x^1 + \varepsilon\left[\mu^{22}(t) + \varepsilon\nu^{22}(t)\right]x^2 . \tag{4.61}$$

Components of matrices μ^{ij}, ν^{ij} are determined by coefficients of the system (4.60)

$$\begin{aligned}
\mu^{11}_{jk}(t) &= \lambda_j^{-1}\xi_{jk}(t)\sin\lambda_j t\cos\lambda_k t ,\\
\mu^{12}_{jk}(t) &= \lambda_j^{-1}\xi_{jk}(t)\sin\lambda_j t\sin\lambda_k t ,\\
\mu^{21}_{jk}(t) &= -\lambda_j^{-1}\xi_{jk}(t)\cos\lambda_j t\cos\lambda_k t ,\\
\mu^{22}_{jk}(t) &= -\lambda_j^{-1}\xi_{jk}(t)\cos\lambda_j t\sin\lambda_k t .
\end{aligned} \tag{4.62}$$

It follows from Theorem A.14 that for $\varepsilon \to 0$ a solution $\left(x^1(t,\varepsilon), x^2(t,\varepsilon)\right)$ of the system (4.61) weakly converges to a $2n$–dimensional diffusion process $\left(x_0^1(\tau),\ x_0^2(\tau)\right)$, $\tau = \varepsilon^2 t$, satisfying equations

$$\begin{aligned}
dx_0^1 &= \left(b^{11}x_0^1 - \beta x_0^1 + b^{12}x_0^2\right)d\tau + \sigma^{11}dw^1 + \sigma^{12}dw^2 ,\\
dx_0^2 &= \left(b^{21}x_0^1 - \beta x_0^2 + b^{22}x_0^2\right)d\tau + \sigma^{12}dw^1 + \sigma^{22}dw^2 .
\end{aligned} \tag{4.63}$$

Calculating diffusion coefficients, we get

$$\begin{aligned}
A^{11}_{jk} &= x_j^1\alpha_{jk}x_k^1 + x_j^2\delta_{jk}x_k^2 ,\\
A^{12}_{jk} &= x_j^1\alpha_{jk}x_k^2 - x_j^2\delta_{jk}x_k^1 ,\\
A^{21}_{jk} &= -x_j^1\delta_{jk}x_k^2 + x_j^2\alpha_{jk}x_k^1 ,\\
A^{22}_{jk} &= x_j^1\delta_{jk}x_k^1 + x_j^2\alpha_{jk}x_k^2 ,\\
A^{11}_{jj} &= \sum_{\substack{k=1\\ j\neq k}}^{n}{}' \left[\left(x_k^1\right)^2 + \left(x_k^2\right)^2\right]q_{jk} + \alpha_{jj}\left(x_j^1\right)^2 + \eta_{jj}\left(x_j^2\right)^2\\
A^{22}_{jj} &= \sum_{\substack{k=1\\ j\neq k}}^{n}{}' \left[\left(x_k^1\right)^2 + \left(x_k^2\right)^2\right]q_{jk} + \eta_{jj}\left(x_j^1\right)^2 + \alpha_{jj}\left(x_j^2\right)^2\\
A^{12}_{jj} &= A^{21}_{jj} = \left(\alpha_{jj} + \eta_{jj}\right)x_j^1x_j^2 , \qquad j \neq k , \qquad j,k = 1, \dots, n.
\end{aligned} \tag{4.64}$$

Here coefficients α_{jj}, α_{jk}, q_{jk} are expressed by relations (4.50)

$$\delta_{jk} = \frac{1}{8\lambda_j\lambda_k}\left[S^{\xi}_{jk.kj}(\lambda_j+\lambda_k)+S^{\xi}_{jk.kj}(\lambda_j-\lambda_k)\right],$$

$$\eta_{jj} = \frac{1}{4\lambda_j^2}\left[S^{\xi}_{jj}(0)+\frac{1}{2}S^{\xi}_{jj}(2\lambda_j)\right]. \tag{4.65}$$

The main characteristic of the solution of program control problems is an expected value of the Cauchy matrix

$$h_\varepsilon(t,t_0) = MK_\varepsilon(t,t_0). \tag{4.66}$$

It follows from Theorem A.14 that for $\varepsilon \to 0,\ 0 \le t-t_0 \le T\varepsilon^{-2}$,

$$\left|h_\varepsilon(t,t_0)-h_{0\varepsilon}(t,t_0)\right| \le C\varepsilon,$$

where, owing to (4.59),

$$h_{0\varepsilon}(t,t_0) = F_c(t)M^1(\tau,\tau_0)+F_s(t)M^2(\tau,\tau_0),$$

$$M^j(\tau,\tau_0) = Mx_0^j(\tau), \quad M^j(\tau_0,\tau_0) = x_0^j(\tau_0) = q^j. \tag{4.67}$$

In its turn, from (4.43) we get

$$dM^1/d\tau = \left(b^{11}-\beta\right)M^1+b^{12}M^2, \quad M^1(\tau_0,\tau_0) = q^1,$$

$$dM^2/d\tau = \left(b^{22}-\beta\right)M^2+b^{21}M^1, \quad M^2(\tau_0,\tau_0) = q^2. \tag{4.68}$$

Calculating drift coefficients b^{ij}, we will obtain

$$b^{11} = b^{22} = \operatorname{diag}\{\gamma_j\}_{j=1}^n, \quad \gamma_j = \rho_j + \sum_{\substack{k=1\\ j\neq k}}^{n}{}' \alpha_{jk},$$

$$b^{21} = -b^{12} = \operatorname{diag}\{\kappa_j\}_{j=1}^n, \quad \kappa_j = \sum_{k=1}^{n}\kappa_{jk},$$

$$\rho_j = \frac{1}{8\lambda_j^2}\left[S^{\xi}_{jj}(2\lambda_j)-S^{\xi}_{jj}(0)\right], \tag{4.69}$$

$$\kappa_{jk} = \frac{1}{4\lambda_j\lambda_k}\left[Z^{\xi}_{jk.kj}(\lambda_j+\lambda_k)-Z^{\xi}_{jk.kj}(\lambda_j-\lambda_k)\right],$$

$$Z^{\xi}_{jk.kj}(\omega) = \int_0^\infty K^{\xi}_{jk.kj}(t)\sin\omega t\,dt.$$

Thus, the system (4.68) is divided into n independent subsystems

$$dm_j^1/d\tau = (\gamma_j - \beta_j)m_j^1 - \kappa_j m_j^2,$$

$$dm_j^2/d\tau = \kappa_j m_j^1 + (\gamma_j - \beta_j)m_j^2,$$

$$m_j^1(\tau_0) = -\lambda_j^{-1}\sin\lambda_j t_0, \quad m_j^2(\tau_0) = \lambda_j^{-1}\cos\lambda_j t_0,$$

$$M^k = \text{diag}\{m_j^k\}, \quad j = 1, \ldots, n, \quad k = 1,\ 2, \tag{4.70}$$

that gives

$$m_j^1 = -\lambda_j^1 \exp[(\gamma_j - \beta_j)(\tau - \tau_0)]\sin[\lambda_j t_0 + \kappa_j(\tau - \tau_0)],$$

$$m_j^2 = \lambda_j^2 \exp[(\gamma_j - \beta_j)(\tau - \tau_0)]\cos[\lambda_j t_0 + \kappa_j(\tau - \tau_0)]. \tag{4.71}$$

Finally, substituting (4.71) into (4.67) and accounting for the form of the matrices $F_c(t)F_s(t)$, we get that

$$h_{0\varepsilon}(t,t_0) = \text{diag}\{h_{0\varepsilon}^j(t - t_0)\}_{j=1}^n,$$

$$h_{0\varepsilon}^j(t - t_0) = \lambda_j^{-1}\exp[(\gamma_j - \beta_j)(\tau - \tau_0)]\sin[(\lambda_j - \varepsilon^2\kappa_j)(t - t_0)]. \tag{4.72}$$

Eqs. (4.72) determine (in the first approximation) an expected value of elements of the Cauchy matrix for a linear parametrically disturbed system. From (4.72) we get averaged necessary stability conditions: $\gamma_j < \beta_j$. In particular, for a system with one degree of freedom we will have

$$\frac{1}{8\lambda_j^2}\left[S^{\xi}(2\lambda_j) - S^{\xi}(0)\right] < \beta. \tag{4.73}$$

It can be easily shown that conditions (4.73) are weaker than (4.56). If they fail then $|Mx(t)| \to \infty$ for $t \to \infty$.

Suppose that correlation functions of disturbances $\xi_{jk}(t,\omega)$ are limited by $e^{-\rho t}$, and the correlation time is considerably smaller than the oscillation period, and $\rho/2\Omega_j >> 1$, $j = 1, 2,\ \ , n$. It can be easily shown that in this case

$$Z_q^{\xi}(\Omega_j \pm \Omega_k) << S_q^{\xi}(\Omega_j \pm \Omega_k), \quad q = jk, kj,$$

and in the relations (4.70) – (4.72) it can be considered that $\kappa_j = 0$.

4.3 Stationary Regimes in Systems with Random Disturbances

4.3.1 General Definitions

In Section 4.2 behavior of the disturbed system for small $\varepsilon \to 0$ for the arbitrary large but finite, $\propto \varepsilon^{-2}$, time interval was analyzed. In applications to problems of the oscillation theory and the control theory, information on asymptotic behavior of the system for small finite $\varepsilon \le \varepsilon_0$, but for $t \to \infty$, is essential. It is linked, in the first place, with problems of existence of stationary motions and with an estimate of functionals for a stationary motion.

Let us introduce some definitions.

A random process $\xi(t)$, $-\infty < t < \infty$, with values from R_l is called *stationary* (in a narrow sense) when for any finite set of numbers $t_1,\dots,t_n$ a joint distribution of random values $\xi(t_1+h),\dots, \xi(t_n+h)$ does not depend on *h*. If in this definition an arbitrary value *h* is replaced with $h = k\theta$ ($k = \pm 1,\ \pm 2$), then we get a definition for a *θ-periodic random process* (or a *generalized θ-periodic random process*).

In other words, any finite–dimensional distribution function $P(t_1,\dots,t_n; A_1,\dots,A_n)$ does not depend on $t_1,\dots,t_n$ for a stationary (in a narrow sense) process and is *T*-periodic with respect to all arguments $t_1,\dots,t_n$ for *a generalized T-periodic process*. Analogously, a *generalized nearly-periodic process* can be defined.

Existence conditions and properties of periodic and stationary solutions of differential equations with a random right-hand part are studied in detail in [129].

A practical importance have only such solutions which possess a definite stability with respect to initial conditions. A periodic (nearly-periodic, stationary) solution $\bar{x}(t)$ of the equation

$$\frac{dx}{dt} = F(t, x, \xi(t)) \tag{4.74}$$

is called *stable* in a respective sense for initial conditions from some domain *K* when for any random values $a \in K$ a solution $X^{s,a}(t)$ of Eq. (4.74) satisfying initial conditions

$$X^{s,a}(s) = a$$

converges for $s \to -\infty$ to $\bar{x}(t)$ in a respective sense.

In analogy with deterministic systems, a different from zero stable solution

$\bar{x}(t)$ can be called a *stationary* solution.

A stationary (in a narrow sense) Markov process is called *homogeneous.* A homogeneous Markov process is generated by a stochastic differential equation

$$dx = b(x)dt + \sigma(x)dw \tag{4.75}$$

with coefficients, independent of t. The properties of homogeneous Markov process are thoroughly studied in [45, 46, 129]. The conditions of existence and stability of homogeneous and non-homogeneous Markov processes can be expressed in terms of Lyapunov functions [68, 129].

4.3.2
Convergence of Disturbed Motion to a Limit Homogenous Diffusion Process

In Section 4.2, a closeness of solutions of disturbed and limit diffusion systems over a finite (with respect to τ) time interval is stated. Let now suppose that the limit diffusion equation (4.35) has a stationary solution $\bar{x}_0(\tau)$. Let us obtain conditions, under which a solution $x_\varepsilon(\tau)$ converges for $\tau \to \infty$ to the homogeneous Markov process $\bar{x}_0(\tau)$.

This problem was discussed for the first time in [153]. The system of equations

$$dx_\varepsilon/d\tau = \varepsilon^{-1}F(x_\varepsilon,\xi_\varepsilon) + G(x_\varepsilon,\xi_\varepsilon), \qquad x_\varepsilon \in R_n \tag{4.76}$$

was studied, where F, G are sufficiently smooth functions, $MF(x,\xi_\varepsilon) = 0$, $\xi_\varepsilon(\tau) = \xi(\tau/\varepsilon^2)$ is a stationary l–dimensional Markov process.

Considering that conditions of Theorem A.14 are satisfied, it can be stated that for $0 \le \tau \le T$ and for $\varepsilon \to 0$ a solution of Eq. (4.76) weakly converges to a diffusion process $x_0(\tau)$ which is a solution of the stochastic differential equation

$$dx_0 = b(x_0)d\tau + \sigma(x_0)dw, \tag{4.77}$$

coefficients of which are calculated according to (4.36).

It was supposed that there exists a stationary solution $\bar{x}_0(\tau)$ of Eq. (4.76) with a corresponding generating operator

$$\mathcal{L} = b'(x)\frac{\partial}{\partial x} + \frac{1}{2}\operatorname{Tr}\sigma(x)\sigma(x)\frac{\partial^2}{\partial x^2}, \tag{4.78}$$

and there exist a sufficiently smooth Lyapunov function $V(x) \ge 0$, $V(x) \to \infty$ for

$|x| \to \infty$ and a constant $\gamma > 0$, such that

$$\mathcal{L}V(x) \le -\gamma V(x) \quad \text{for} \quad |x| \to \infty. \tag{4.79}$$

In [153] it was shown that when these conditions are satisfied then there exists a stationary distribution of the process $v_\varepsilon(\tau) = \{x_\varepsilon(\tau), \xi_\varepsilon(\tau)\}$, and a finite-dimensional probability density $p_\varepsilon(x)$ of the process $x_\varepsilon(\tau)$ for $\varepsilon \to 0$ weakly converges to a probability density $p_0(x)$ of the process $\bar{x}_0(\tau)$.

For applications it is important to obtain an analogous result for a case when the process $\xi_\varepsilon(\tau)$ is not necessarily a Markov stationary process, and the process $v_\varepsilon(\tau)$ does not have a stationary distribution.

Besides, in [153] the question of closeness of the processes $x_\varepsilon(\tau)$ and $\bar{x}_0(\tau)$ for arbitrary initial conditions for $x_\varepsilon(\tau)$ was not discussed. And this problem is very important for applications, since a practical meaning have only the motions which are stable with respect to a disturbance of initial conditions.

More general results on an approximation of stationary motions are given in [179]. The main suggestions concern the properties of the processes $x_\varepsilon(\tau)$ and $x_0(\tau)$ and need a concretization.

Theorem 4.4. *Let processes* $x_\varepsilon(\tau)$ *and* $x_0(\tau)$ *satisfy the following conditions:*

1) *the diffusion equation* (4.77) *has a unique solution* $x_0(\tau) = X_0^{s,a}(\tau)$ *for any initial conditions* $X_0^{s,a}(\tau) = a \in K$;

2) *there exists a unique stationary solution* $x_0(\tau)$ *of Eq. (4.77), i.e.,*

$$\lim_{s \to -\infty} Mf\left(X_0^{s,a}(\tau)\right) = Mf\left(\bar{x}_0(\tau)\right) \tag{4.80}$$

for any continuous function $f(x)$ *uniformly with respect to* $a \in K$ *for any compact* $K \subset R_n$;

3) *for* Eqs. (4.76), (4.77) *conditions of* Theorem A.14 *are satisfied*;

4) *for* $0 < \varepsilon \le \varepsilon_0$, $\tau \ge 0$ *a succession of processes* $x_\varepsilon(\tau)$ *is weakly compact.*

Then for any continuous function $f(x)$ *and any* $\delta > 0$, $T < \infty$ *there can be found values* $\tau_0 < \infty$, $\varepsilon_0 > 0$ *such that for any* $\tau \ge \tau_0$, $\Delta \le T$, $0 < \varepsilon \le \varepsilon_0$

$$\left|Mf\left(x_\varepsilon(\tau + \Delta)\right) - Mf\left(\bar{x}_0(\Delta)\right)\right| < \delta. \tag{4.81}$$

If the function $f(x)$ is sufficiently smooth then from conditions 2) – 4) of Theorem follows that for small ε the statement (4.81) is equivalent to the relation [comp. (A.118)]

$$\lim_{x\to-\infty}\left|Mf\left(x_\varepsilon(\tau)\right)-f_0\right|\le C\varepsilon\,,\qquad C=C(f)=\mathrm{const}(\varepsilon). \tag{4.82}$$

Here

$$f_0=Mf\left(\bar{x}_0(\tau)\right)=Mf\left(\bar{x}_0(\tau_0)\right) \tag{4.83}$$

for all $\tau\ge\tau_0$.

Let us formulate conditions of Theorem 4.4 in the form of requirements to coefficients of the initial and limit equations.

Existence conditions of a limit stationary distribution which provide a fulfillment of requirements 1) and 2) are shown, for instance, in [68, 129]. It is shown [129, Sections 4.4, 4.5] that the conditions 1) and 2) are satisfied when in some bounded domain $U\subset R_n$

A.1. Coefficients $\sigma(x)$, $b(x)$ of the limit diffusion equation (4.77) satisfy the conditions (4.8) and the smallest eigenvalue of the diffusion matrix $A=\sigma\sigma'$ is bounded from below by some positive number.

A.2. There exists two-time differentiable non-negative function $V(x)$ for which

$$\mathcal{L}V(x)\le -1 \qquad \text{for} \qquad x\in R_n\setminus U$$

holds, where $\mathcal{L}$ is a generating operator (4.78) corresponding to the process $x_0(\tau)$. Considering $V(x)=(x,x)$, we get, in particular, that Condition A.2 is satisfied when for sufficiently large $|x|$

$$\mathrm{Tr}\,A(x)+2\left(x,b(x)\right)<-1\,. \tag{4.84}$$

Thus, the suggestions 1) and 2) hold true when the coefficients of the limit equation (4.77) satisfy Eqs. (1.8), (3.11).

Physical meaning of Conditions A.1, A.2 was explained in [129]. Condition A.1 means that there exists a stationary solution of the diffusion equation other than zero; Condition A.2 provides stability of this solution with respect to probability.

Conditions 3), 4) are related to the disturbed system. Condition 3) is satisfied when the coefficients of the disturbed system meet the requirements (A) and (B) of Theorem A.14.

Conditions of weak compactness for the sequence $\{x_\varepsilon(\tau)\}$ on an infinite interval are named in [179].

Consider firstly that $\xi(t)$ is a bounded continuous on the right stationary random process satisfying a condition of uniform strong mixing. Then the sequence $\{x_\varepsilon(\tau)\}$ is weakly compact when

A.3. Coefficients F, G satisfy conditions of smoothness of Theorem A.14 and for $|x|\to\infty$, $|\xi|\le K_0$

$$|F(x,\xi)| + |G(x,\xi)| \le K(1+|x|).$$

A.4. There exists positively determined three-times differentiable function $V(x)$ such that for $|\xi| \le K_0$, $x \in R_n$

$$\left|[V_x(x)F(x,\xi)]_x F(x,\xi)\right| \le K(1+V(x)),$$

$$\left|\left\{[V_x(x)F(x,\xi)]_x F(x,\xi)\right\}_x U(x,\xi)\right| \le K(1+|\mathrm{L}V(x)|).$$

Here $U = G, F$; L is the operator (4.78).

A.4′. Condition A.4 is satisfied when derivatives F_x, F_{xx} are bounded for all $x \in R_n$ uniformly with respect to ξ for $|\xi| \le K_0$. Then a function $V(x) = (x, Px)$ can be taken instead of $V(x)$, where P is a positively determined matrix.

It can be easily proved that statements of Theorem 4.4 hold true when $\xi(t)$ is not a stationary process but a generalized periodic or quasi-periodic process of the form

$$\xi(t) = T(t)\tilde{\xi}(t), \tag{4.85}$$

where $T(t)$ is a matrix, components of which are bounded periodic or uniformly quasi-periodic processes, $\tilde{\xi}(t)$ is a bounded continuous on the right process satisfying a condition of uniform strong mixing.

At last, if the process $\xi(t)$ is not a stationary process but a Markov one with a normal distribution, then Theorem 4.4 also holds true. In this case Condition A.3 is replaced with a condition of a linear growth of the coefficients with respect to x uniformly with respect to ξ in any bounded domain, and Condition A.4 is replaced with Eq. (4.79).

Let explain these conditions, stating existence conditions and estimating a mean-square oscillation amplitudes in a linear system.

Let, for instance, dynamics of the system be described by Eq. (4.57): $H(x,\dot{x}) = 2\beta\dot{x}$. Let reduce (4.57) be means of replacement (4.59) to the standard form

$$\begin{aligned} \dot{x}^1 &= \varepsilon[\mu^{11}(t) + \varepsilon v^{11}(t)]x^1 + \varepsilon[\mu^{12}(t) + \varepsilon v^{12}(t)] + \varepsilon z^1(t), \\ \dot{x}^2 &= \varepsilon[\mu^{21}(t) + \varepsilon v^{21}(t)]x^1 + \varepsilon[\mu^{22}(t) + \varepsilon v^{22}(t)] + \varepsilon z^2(t), \end{aligned} \tag{4.86}$$

where coefficients μ, v are calculated according to (4.60), (4.62) and

$$z_j^1 = -\lambda_j^{-1}\zeta_j(t)\sin\lambda_j t, \qquad z_j^2 = \lambda_j^{-1}\zeta_j(t)\cos\lambda_j t. \tag{4.87}$$

It is supposed that $\xi_{ij}(t)$, $\zeta_j(t)$ are stationary and stationary–linked processes with a zero mean, satisfying mentioned above conditions. And the system (4.86) satisfies Conditions A.3, A.4 and the conditions 3), 4) of Theorem are satisfied. The limit diffusion equation corresponding to the system (4.86) has the form [comp. (4.63)]

$$dx_0^1 = \left(b^{11}x_0^1 - \beta x_0^1 + b^{12}x_0^2\right)d\tau + s^{11}dw^1 + \sigma^{12}dw^2,$$
$$dx_0^2 = \left(b^{21}x_0^1 - \beta x_0^2 + b^{22}x_0^2\right)d\tau + \sigma^{21}dw^1 + s^{22}dw^2. \tag{4.88}$$

Here $s^{pp}\left(s^{pp}\right)' = A^{pp} + d$, $d = \operatorname{diag}\left\{d_{jj}\right\}$, $p = 1,2$; $j = 1, \ldots, n$; coefficients b^{pq}, σ^{pq}, A^{pp} are determined by Eqs. (4.64), (4.69), coefficients d_{jj} are determined by Eq. (4.50).

Obviously, the eigenvalues p_j of the diffusion matrix for sufficiently small $|x|$ are bounded by the values d_{jj}, i.e., Condition A.1 holds. The condition (4.84) can be also simply presented in terms of coefficients of Eq. (4.88). Thus, a stationary solution $\left(\bar{x}_0^1, \bar{x}_0^2\right)$ of Eqs. (4.88) exists, but $M\bar{x}_0^1 = M\bar{x}_0^2 = 0$.

A mean-square value of the oscillation amplitude $z_{\varepsilon j} = R_j^2 = \left(x_j^1\right)^2 + \left(x_j^2\right)^2$, where x_j^1, x_j^2 are components of the solution vector of Eq. (4.86), remains a more informative characteristic. Building equations for coefficients $z_{\varepsilon j}$, it is easily seen that they also satisfy Conditions A.3, A.4.

Let $\left(R_j^0\right)^2 = \left(x_{0j}^1\right)^2 + \left(x_{0j}^2\right)^2 = z_j$. Processes z_j satisfy Eq. (4.52), and the vector of mean-square oscillation amplitudes of the stationary solution can be found as a stationary solution of Eq. (4.53)

$$\overline{M}_z = -2Q^{-1}d, \quad \overline{M}_z = \left(M\bar{z}_1, \ldots, M\bar{z}_n\right). \tag{4.89}$$

In particular, for a system with one degree of freedom we have [comp. (4.55)]

$$M_z = \frac{S^\zeta(\lambda)}{2\lambda^2\left[\beta - S^\zeta(2\lambda)/4\lambda^2\right]}. \tag{4.90}$$

Correctness of the conditions of Theorem for Eq. (4.53) is easily proved. Eq. (4.84) coincides with the condition of asymptotic stability of the system (4.53). Thus, for stationary values $\bar{z}_{\varepsilon j}$, $\bar{z}_j$ an estimate

$$\overline{M}_z = -2Q^{-1}d, \quad \left|M\bar{z}_{\varepsilon j} - M\bar{z}_j\right| \le C\varepsilon \tag{4.91}$$

holds.

4.4 Oscillations of Vibroimpact Systems at Random Disturbance

Problems of stochastic dynamics of vibroimpact systems arise in study of oscillations of machines with impact elements, in analysis of mechanical and controlled systems with clearances and limiters. In analysis of stochastic systems it is usually necessary to construct probability characteristics of coordinates and velocities of striking elements and to estimate stability of disturbed movement [16, 21, 22, 47, 50, 74, 78, 97].

Exact relations for probability density and dispersion of coordinates and velocities were obtained only for a system with symmetric limiters, excited by a white noise, for an elastic impact [16, 21, 22, 47, 50, 51]. In this particular case, an exact solution of the Fokker–Planck–Kolmogorov (FPK) equation describing the probability density is constructed. For more complex systems with external excitation other than the white noise, a solution was constructed with the use of the method of statistical linearization [16], which is broadly used in applications, but does not allow the estimation of obtained results.

In [47] asymptotic methods of investigation of stochastic vibroimpact systems, close to conservative ones, were developed. With the help of the method of non-smooth transformations [57] the system was reduced to the standard form, and the FPK equation characterizing the oscillation energy distribution was formulated. The main results were obtained for system with the white–noise excitation: the formulation of the FPK equation for another type of excitation is complicated.

An advantage of such an approach is a principle possibility of the analysis of non-quasi-isochronous systems. But relations for the drift and diffusion coefficients for this case are very cumbersome, and it is possible to find an analytical solution of the FPK equation and respective moment characteristics practically only for a quasi-isochronous system.

The transformation (3.108) allows the construction of statistic characteristics of an impact impulse. And the transformation to the standard form is easier than in the method of non-smooth transformations.

Let us study in detail a quasi-isochronous system $(\Delta = 0)$ with one degree of freedom and one-sided limiter. Results can be easily generalized for systems with more complex structure and with two-sided limiters.

Let dynamics of a system be described by the equation

$$\ddot{x}+\Omega^2 x=\varepsilon g_1(x,\dot{x})\xi(t)+\varepsilon g_2(x,\dot{x}), \tag{4.92}$$

and by a condition of an impact against the one-sided limiter

$$x=0, \quad \dot{x}_+=-(1-\varepsilon^2 r)\dot{x}_- . \tag{4.93}$$

Let $\xi(t)$ be a stationary random process satisfying conditions (A) of Theorem

A.14, functions g_1, g_2 are supposed to be sufficiently smooth with respect to x, $\dot{x}$; ε is a small parameter. An impact condition $x = 0$ means that a generating system is isochronous for $\varepsilon = 0$, its eigenfrequency is 2Ω.

Supposing that an excitation is close to a one-impact π/Ω – periodic one, we will introduce the replacement of variables (3.122)

$$x = -J\chi(\theta), \quad \dot{x} = -2\Omega J\chi_\theta(\theta), \quad \theta = 2\Omega(t-\varphi), \tag{4.94}$$

where $\chi(\theta)$ is a periodic Green's function of the generating system, determined by the relation

$$\chi(\theta) = \frac{1}{2\Omega}\sin\frac{\theta}{2}, \quad 0 < \theta < 2\pi. \tag{4.95}$$

The replacement (4.94), (4.95) transforms the system (4.92), (4.93) to the standard form (3.123). In order to avoid singularities of the kind J^{-1}, we will consider

$$J = e^y. \tag{4.96}$$

In analogy with (3.123) we get

$$\begin{aligned} \dot{y} &= -\varepsilon^2 2\Omega r\delta^2\pi(\theta) + \varepsilon G_{11}(y,\theta)\xi(t) + \varepsilon^2 G_{12}(y,\theta), \\ \dot{\varphi} &= \varepsilon G_{21}(y,\theta)\xi(t) + \varepsilon^2 G_{22}(y,\theta). \end{aligned} \tag{4.97}$$

Here $\delta^{2\pi}(\theta)$ is a 2π-periodic Dirac δ-function,

$$\begin{aligned} G_{1j} &= -8\Omega e^{-y} g_j\left(e^{-y}\chi, -2\Omega e^{-y}\chi_\theta\right)\chi_\theta, \\ G_{2j} &= -4e^{-y} g_j\left(e^{-y}\chi, -2\Omega e^{-y}\chi_\theta\right)\chi, \end{aligned} \tag{4.98}$$

discontinuity moments t_k are determined by relations

$$\theta(t_k) = 2\pi k, \quad k = 0, 1, \ldots \tag{4.99}$$

The replacement (4.94) is non-smooth, and functions G_{jl} has discontinuities of the first kind with respect to θ in points (4.99). Hence, the 2π-periodic Dirac δ-function should appear in derivatives $\partial G_{jl}/\partial\varphi = -2\Omega\,\partial G_{jl}/\partial\theta$.

Working in the same way as in Section A.6, it can be easily shown that Theorem A.13 holds true when conditions (B) of Theorem A.14 for the system (4.97) are satisfied only with respect to y. Estimates for discontinuous variables are constructed in the same way as for a deterministic case. Another transformation method for systems with discontinuous coefficients was proposed in [179]. The discontinuous coefficients were approximated by some converging sequence of smooth functions in a prove of respective estimates, and then a correctness of the inverse transition with respect to a parameter characterizing the approximating sequence of smooth functions was proved.

Let us apply conclusions of Section A.6 to vibroimpact systems of the type (4.97). We suppose that a random disturbance ξ satisfies conditions (A), and coefficients G_{jl} satisfy conditions (B) of Theorem A.14 (with respect to slow variable y). Let us analyze some particular cases.

1) *Parametric random excitation.* Dynamics of a system is described by the equation

$$\ddot{x}+\Omega^2(1+\varepsilon\xi(t))x+\varepsilon^2(2\beta\dot{x}+\Omega^2\Delta)=0\,, \tag{4.100}$$

and by impact conditions (4.93). Here $\varepsilon^2\Delta$ is a clearance (press fit), a coordinate x is reduced to a zero clearance (press fit).

Reduce the system (4.100), (4.93) to the standard form (4.97). Coefficients G_{jl} can be written in a form accounting for a direct dependence on time:

$$\begin{aligned}\dot{y}&=-4\varepsilon\left[\Omega^2\chi\chi_t\xi(t)+\varepsilon\left(2\beta\chi_t-\Omega^2\Delta e^{-y}\right)\chi_t\right]-\varepsilon^2 2\Omega r\delta^{2\pi}(\theta)\,,\\ \dot{\varphi}&=-4\varepsilon\left[\Omega^2\chi^2\xi(t)+\varepsilon\left(2\beta\chi_t-\Omega^2\Delta e^{-y}\right)\chi\right],\end{aligned} \tag{4.101}$$

where $\chi=\chi[2\Omega(t-\varphi)]$, $\chi_t=2\Omega\chi_\theta$, $\theta=2\Omega(t-\varphi)$.

In analogy with (4.41) we denote

$$\begin{aligned}&F^1=-4\Omega^2\zeta_1\xi\,,\qquad F^2=-4\Omega^2\zeta_2\xi\,,\\ &G^1=-4\left[2\beta\zeta_3-\Omega^2\Delta e^{-y}\chi_t\right],\qquad G^2=-4\left[2\beta\zeta_1-\Omega^2\Delta e^{-y}\chi\right],\end{aligned} \tag{4.102}$$

where the arguments of functions χ and ζ_j are omitted, and

$$\zeta_1=\chi\chi_t\,,\qquad \zeta_2=\chi^2\,,\qquad \zeta_3=\chi_t^2 \tag{4.103}$$

for $\chi=\chi[2\Omega(t-\varphi)]$, $\zeta_j=\zeta_j[2\Omega(t-\varphi)]$, and in the interval $0<t<T=\pi/\Omega$

$$\chi(2\Omega t)=\frac{1}{2\Omega}\sin\Omega t\,,\qquad \chi_t(2\Omega t)=\frac{1}{2}\cos\Omega t\,. \tag{4.104}$$

Owing to (4.103), (4.104), the functions $\zeta_j(2\Omega t)$, $j=1,2,3$, for $-\infty<t<\infty$ have the form

$$\begin{aligned}&\zeta_1(2\Omega t)=\frac{1}{8\Omega}\sin 2\Omega t\,,\qquad \zeta_2(2\Omega t)=\frac{1}{8\Omega^2}(1-\cos 2\Omega t)\,,\\ &\zeta_3(2\Omega t)=\frac{1}{8}(1+\cos 2\Omega t)\,,\end{aligned}$$

i.e., the functions ζ_j are continuous for $-\infty<t<\infty$.

Below it is considered that $\xi(t)$ is a stationary random process with a correlation function $K_\xi(t)$.

It can be easily shown that the limit diffusion equation for y gets the form

$$dy^0 = \left(-rT^{-1} - \beta + \nu\right)d\tau + \sigma dw, \quad T = \pi/\Omega. \tag{4.105}$$

The term rT^{-1} is formed by means of averaging of the T-periodic succession of delta functions, the coefficients ν and σ are calculated according to Eqs. (4.34)

$$\nu = \frac{16\Omega^4}{T}\int_0^T ds \int_{-\infty}^{0} \frac{\partial \zeta_1[2\Omega(s-\varphi)]}{\partial\varphi}\zeta_2[2\Omega(t+s-\varphi)]K_\xi(t)dt = \frac{\Omega^2}{8}K_\xi(2\Omega), \tag{4.106}$$

$$\sigma^2 = \frac{16\Omega^4}{T}\int_0^T ds \int_{-\infty}^{\infty} \zeta_1[2\Omega(s-\varphi)]\zeta_1[2\Omega(t-\varphi)]K_\xi(t-s)dt = \frac{\Omega^2}{8}K_\xi(2\Omega).$$

Here $S_\xi(\lambda)$ is a spectral density of the process $\xi(t)$.

In the most cases, a mean-square value of the impulse $\mu = MJ^2$ is of interest. Denoting $z = e^{2y}$, we will obtain that the function z for $\varepsilon \to 0$ weakly converges to the process z^0 – a solution of the diffusion equation

$$dz^0 = \left[-2\left(\beta + \frac{r\Omega}{\pi}\right) + \frac{1}{2}\Omega^2 S_\xi(2\Omega)\right]z^0 d\tau + 2z^0\sigma dw. \tag{4.107}$$

Eq. (4.107) is obtained from (4.105) by means of the application of the Itô's formula (4.9) to the function $z^0 = \exp\left(2y^0\right)$. The mean-square value of the impulse, respectively, $MJ^2 \to \mu^0$, where μ^0 is the solution of the equation

$$\frac{d\mu^0}{d\tau} = \left[-2\left(\beta + \frac{r\Omega}{\pi}\right) + \frac{1}{2}\Omega^2 S_\xi(2\Omega)\right]\mu^0. \tag{4.108}$$

It follows from (4.108) that for

$$\frac{\Omega^2}{2}S_\xi(2\Omega) > 2\left(\beta + \frac{r\Omega}{\pi}\right) \tag{4.109}$$

the regime under study is mean-square unstable; if

$$2\beta < \frac{\Omega^2}{2}S_\xi(2\Omega) < 2\left(\beta + \frac{r\Omega}{\pi}\right) \tag{4.110}$$

then the energy dissipation at the impact stabilizes the system motion.

2) *The motion of a vibroimpact system at a random–force excitation.* An equation of the system motion between impacts has the form

$$\ddot{x} + \Omega^2 x + \varepsilon^2\left(2\beta\dot{x} + \Omega^2\Delta\right) = \varepsilon\xi(t), \tag{4.111}$$

the impact conditions (4.93) hold. Eqs. (4.111) are reduced to the standard form

$$\dot{y} = -4\varepsilon\left\{e^{-y}\chi_t\xi(t) + \varepsilon\left[2\beta\chi_t + \Omega^2\Delta e^{-y}\right]\chi_t\right\} - 2\varepsilon^2\Omega r\delta^{2\pi}(\theta), \tag{4.112}$$
$$\dot{\varphi} = -4\varepsilon\left\{e^{-y}\chi\xi(t) + \varepsilon\left[2\beta\chi_t + \Omega^2\Delta e^{-y}\right]\chi\right\},$$

by the replacement (4.94), (4.95).

Owing to (4.102), denote

$$F^1 = -4e^{-y}\chi_t\left[2\Omega(t-\varphi)\right]\xi(t),$$
$$F^2 = -4e^{-y}\chi\left[2\Omega(t-\varphi)\right]\xi(t), \tag{4.113}$$

the functions G^1, G^2 keep the form (4.102).

The diffusion equation for the limit process y^0 gets the form

$$dy^0 = \left(-rT^{-1} - \beta + \gamma\exp\left(-2y^0\right)\right)d\tau + \sigma\exp\left(-2y^0\right)dw. \tag{4.114}$$

The coefficients γ, σ are calculated according to Eqs. (4.34)

$$\gamma = \frac{16}{T}\int_0^T ds\int_{-\infty}^0\left\{-\chi_s\left[2\Omega(s-\varphi)\right]\chi_t\left[2\Omega(t+s-\varphi)\right]\right.$$
$$\left. + \chi_{s\varphi}\left[2\Omega(s-\varphi)\right]\chi\left[2\Omega(t+s-\varphi)\right]\right\}K_\xi(t)dt = -\gamma_1 + \gamma_2, \tag{4.115}$$

$$\sigma^2 = \frac{16}{T}\int_0^T ds\int_{-\infty}^{\infty}\chi_s\left[2\Omega(s-\varphi)\right]\chi_t\left[2\Omega(t-\varphi)\right]K_\xi(t-s)dt.$$

Replace $\chi(2\Omega t)$ by relations

$$\chi(2\Omega t) = \frac{1}{\pi\Omega} + \frac{2}{\pi\Omega}\sum_{k=1}^{\infty}\frac{\cos 2k\Omega t}{1-4k^2},$$
$$\tag{4.116}$$
$$\chi_t(2\Omega t) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{k\sin 2k\Omega t}{1-4k^2},$$

which are true for $-\infty < t < \infty$. Substituting (4.116) into (4.125), we get as a result of obvious transformations

$$\gamma_1 = \gamma_2 = \frac{64}{\pi^2}\sum_{k=1}^{\infty}\frac{k^2}{\left(1-4k^2\right)^2}S_\xi(2k\Omega), \qquad \gamma = 0,$$
$$\tag{4.117}$$
$$\sigma^2 = 2\gamma_1 = \frac{128}{\pi^2}\sum_{k=1}^{\infty}\frac{k^2}{\left(1-4k^2\right)^2}S_\xi(2k\Omega).$$

Thus, Eq. (4.114) is simplified:

$$dy^0 = -\left(\beta + r\Omega/\pi\right)d\tau + \sigma e^{-y^0}dw. \tag{4.118}$$

If the correlation time of the process $\xi(t)$ is small if compared to the oscillation period $T = \pi/\Omega$, i.e.,

$$\int_{-\infty}^{\infty} \left|K_\xi(t)\right| dt \approx \int_{-T}^{T} \left|K_\xi(t)\right| dt ,$$

then, integrating the inner integral in (4.115) with integration limits $-T$ and T and considering $\chi_t(2\Omega t) = 1/2 \cos \Omega t$, $0 < t < T$, we get

$$\sigma^2 \approx 2S_\xi(\Omega). \tag{4.119}$$

Let us now return to the variable $z = J^2 = e^{2y}$. The limit diffusion equation for the function $z^0 = \exp\left(2y^0\right)$ get the form

$$dz^0 = \left[-2\left(\beta + r\Omega/\pi\right)z^0 + 2\sigma^2\right]d\tau + 2z^0 \sigma dw. \tag{4.120}$$

For a mean value $\mu^0 = Mz^0$ we will get a linear equation

$$d\mu^0/d\tau = 2\left[\sigma^2 - \left(\beta + r\Omega/\pi\right)\mu^0\right] \tag{4.121}$$

with a stationary solution

$$\bar{\mu}^0 = 2S_\xi(\Omega)\left(\beta + r\Omega/\pi\right)^{-1}. \tag{4.122}$$

From (4.121), (4.122) follows that the energy dissipation at the impact $(r > 0)$ diminishes the oscillation intensity under random excitation [47]. If there are no dissipation at the impact, $r = 0$ and $\beta < 0$, then the stationary regime is unstable; if $r > |\beta|\pi\Omega^{-1}$ then the one-sided limiter stabilizes the unstable linear system.

5 Some Problems of Optimal Control for Systems with Random Disturbances

Let us now analyze some problems of optimal control for systems that function under uncertain conditions. One of the possible approaches to a formalization of this task is the use of the game theory. In such a case it is supposed that only upper bounds of a level of disturbances are known, and a control is constructed with account for the worst case [3]. The game approach can be utilized also in problems of the oscillation control [29, 76, 192].

A probability approach is found to be important for applications; here all the types of uncertainty are treated as random values with fixed probability or spectral characteristics. The search of an optimal control is then based on an extremum condition for an expected value of some functional, which depends on the control and phase trajectory.

An information that is available to an observer at each moment is important for an optimal control formation.

It is well known [31, 127] that the program control and feedback control are equivalent for deterministic systems, since a system state at any moment t can be determined from an initial state $(t_0, x(t_0))$ and a control $u(s, x(s))$, $s \le t$, used up to the moment t. An observation of the current state $x(t)$ of the system does not provide any new information in comparison with the initial state.

In stochastic systems, the fixed initial state determines only probable characteristics of the trajectory, so the optimal control essentially depends on the information which the observer has for the current moment.

If the state of the system is completely known at the moment t, then the control can be constructed in the form of the feedback. In systems with random excitations, the program control is less effective than the feedback control, since it does not account for the system state at the moment of observation. At the same time, if the information on the system is not available or the control synthesis can not be formed, then the control is constructed in the form of the program.

The principle of dynamic programming, that determines the control synthesis, and the stochastic maximum principle hold true for systems of a sufficiently general form [127, 175, 177]. Still, the main results were obtained for systems, dynamics of which is described by the Itô's equation. This can be explained by the fact that the dynamic programming scheme is effective only in cases when a guided process is the process without aftereffect. If the process is of the white–noise type, then the guided process will be a Markov one, and the dynamic–programming control can be easily formulated in the direct form. For Markov processes, other then the white noise, a solution for the system is a component of a partially observed process of the higher dimension. Therefore, a dimension of the system should be increased for a solution of synthesis problems by means of introduction of a space of sufficient coordinates [90, 121, 122]. An analogous approach which is inevitable for a strict formulation of equations of the dynamic programming considerably complicates a computational procedure.

Let us note that the principle of dynamic programming leaves open the questions of the control synthesis for non-Markov excitations. The stochastic maximum principle is free from such limitations, but the program control construction is reduced, as it will be shown below, to the calculation of moment characteristics of solutions of a system of stochastic equations.

In Sections 4.2, 4.3 it was shown that the solution of the equation with wide-band random excitations converges under certain conditions to the solution of the averaged system of Itô's equations. According to the definition of the weak convergence, the closeness of expected values for functional on trajectories of the initial and averaged systems is provided. It is naturally to suppose that the closeness of mean values of the functionals determines the closeness of optimal controls providing extremum of the functional. The correctness of this suggestion is proved below. It is shown that a solution of the optimal problem is reduced to known optimization algorithms of a functional on trajectories of the limit control system.

In Section 5.1, an algorithm of the program control construction in disturbed systems is given, and the stochastic maximum principle is discussed.

In Section 5.2, problems of the control synthesis in systems with random disturbances, other then the white noise, or, generally speaking, of Markov processes are studied. The main result is a construction of a quasi-optimal control corresponding to a the limit system. Control problems for a stationary motion are given in Section 5.3.

5.1 Program Control in Systems with Random Disturbances

The program control and feedback control are equivalent in deterministic systems, therefore there is no necessity for division of treatment methods for these two types of the control. In stochastic systems, in contrast, the program control depends not on the system state but on some moment characteristics, and thus is less effective.

At the same time, an observer does not always have a complete information about the system. If the system state is inaccessible for a measurement, or the feedback realization is complicated, then the control should be a deterministic function of time. Necessary optimality conditions, analogous to the maximum principle of L.S. Pontryagin, serve for a choice of the optimal control.

There exist various formulations for the stochastic maximum principle, concerning mainly the optimal control construction in systems described by the Itô's equations.

A comparison and discussion of various approaches are given in [144, 149, 150, 165–167, 177, 195, 200].

General conditions of the stochastic maximum principle based on the abstract theory of the optimal control [139] are obtained in [52, 138] (see Appendix, Section A.5). For a deterministic case they are reduced to the known equations of the maximum principle.

5.1.1 Necessary Conditions for Optimal Program Control in Stochastic Systems

We will study below only one program control problem, namely, the Bolza problem. Let us give the necessary conditions of optimality for this problem.

Let dynamics of a system be described by the equation

$$\dot{x} = f\big(t, x, u, \xi(t)\big), \quad x(t_0) = x_0. \tag{5.1}$$

Here $x \in R_n$ is a vector of phase variables, $\xi(t) \in R_l$ is a vector of random disturbances, $u \in U \subset R_m$ is a vector of guiding excitations. A deterministic piecewise continuous control $u(t)$ minimizing the functional

$$\Phi(u) = M\left[\int_{t_0}^{t_f} \varphi\big(t, x(t), u(t)\big)dt + F\big(x(t_f)\big)\right], \tag{5.2}$$

on the trajectories of the system (5.1) should be found. Here $x(t) = X^{t_0, x_0}(t)$, t_f is a fixed moment of the process end. It is supposed that φ and F satisfy the conditions of Section A.5.

The control $u(t)$ is considered to be admissible (as in deterministic systems) if there exists a unique solution of the system (5.1) for $u = u(t)$, and the functional (5.2) is determined.

In analogy with the deterministic case, we will introduce a Hamiltonian

$$H = -\varphi(t, x, u) + \Big(p, f\big(t, x, u, \xi(t)\big)\Big), \tag{5.3}$$

where a Lagrange multiplier p satisfies the equation

$$\dot{p} = -H_x(t, x, u, p), \tag{5.4}$$

and the boundary condition

$$p(t_f) = -F_x\big(x(t_f)\big). \tag{5.5}$$

An optimal control $u_*(t)$ is determined by the stochastic maximum principle [52, 138]

$$u_* = \arg\max_{u \in U} MH(t,x,u,p). \tag{5.6}$$

If the domain U is not bounded, then the equation for the determination of u_* gets the form

$$MH_u(t,x,u,p) = 0. \tag{5.7}$$

Eqs. (5.4), (5.7) are the stochastic analog of the Euler–Lagrange equations; Eq. (5.6) is a necessary condition for the functional extremum.

If the values of functions of phase coordinates at the moment of the process end $\psi(t_f, x(t_f)) = 0$ $(\psi \in R_q)$ are given, then the function $x(t)$ should satisfy two boundary conditions, for $t = t_0$ and $t = t_f$, and there are no boundary conditions in Eq. (5.5). If the problem of the optimal high–speed action is being solved, i.e., the moment t_f is not fixed, then additional relations

$$MH(t_f, x(t_f), u(t_f), p(t_f)) = 0 \tag{5.8}$$

are added to Eqs. (5.6), (5.7).

The high–spedd action problem can be treated in terms of the theory of stochastic differential equations as the minimization problem for the expected value of the time of reaching the given point.

If Eq. (5.1) has a linear dependence on the control u, and the functional has a quadratic dependence

$$f = f_1(t,x,\xi(t)) + d(t)u, \qquad \varphi = \varphi_1(t,x) + u'r(t)u, \tag{5.9}$$

where $r > 0$ is a symmetric matrix of the dimension $m \times m$, then

$$H = -[\varphi_1(t,x) + u'r(t)u] + (p, f_1(t,x(t))) + d(t)u,$$

and the system (5.6) is reduced to the form

$$\dot{p} = -\varphi_{1x}(t,x) - f'_{1x}(t,x,\xi(t))p \tag{5.10}$$

(the prime denotes transpose), and from the condition (5.8) we get

$$u(t) = -\frac{1}{2} r^{-1}(t) d'(t) Mp(t), \tag{5.11}$$

and the problem is reduced to the calculation of the expected value of the adjoint variable $p(t)$.

For a linear system

$$\dot{x} = A(t)x + d(t)u + B(t)\zeta(t), \qquad x(t_0) = a, \tag{5.12}$$

with a quadratic quality criterion

$$\Phi(u) = M\left\{\int_{t_0}^{t_f}\left[x'(t)r_1(t)x(t) - u'(t)r_2(t)u(t)\right]dt + x_3'(t_f)r_3x(t_f)\right\}, \tag{5.13}$$

this problem is solved analytically. Here A, B, d are deterministic matrices of dimensions $n\times n$, $n\times l$, $n\times m$, respectively, $r_1 \geq 0$, $r_2 > 0$ are symmetric matrices of dimensions $n\times n$ and $m\times m$, respectively.

Substituting (5.11) into (5.12) and accounting for (5.10), (5.13), we get

$$\begin{aligned} &\dot{x} = A(t)x + D(t)Mp + B(t)\zeta(t), \quad x(t_0) = a, \\ &\dot{p} = -A'(t)p + 2r_1(t)x, \quad p(t_f) = -2r_3x(t_f), \end{aligned} \tag{5.14}$$

where

$$D(t) = \frac{1}{2}d(t)r_2^{-1}(t)d'(t).$$

The system (5.14) is transformed to a deterministic system of equations for expected values $m_x = Mx$, $m_p = Mp$:

$$\begin{aligned} &\dot{m}_x = A(t)m_x + D(t)m_p + B(t)m_\zeta(t), \\ &\dot{m}_p = -A'(t)m_p + 2r_1(t)m_x, \\ &m_x(t_0) = a, \quad m_p(t_f) = -2r_3m_x(t_f). \end{aligned} \tag{5.15}$$

Here $m_\zeta = M\zeta(t)$. Searching for a solution of the system (5.15) in the form

$$m_p = -R(t)m_x + Q(t), \tag{5.16}$$

we get the known result [69]: a symmetric positively determined matrix R and vector Q satisfy equations:

$$\dot{R} = -2r_1 - A'R - RA + RDR, \tag{5.17}$$

$$R(t_f) = -2r_3, \tag{5.18}$$

and also

$$\dot{Q} = (RD - A')Q + RBm_\zeta, \quad Q(t_f) = 0. \tag{5.19}$$

In its turn, the control (5.11) has the form

$$\begin{aligned} &u(t) = -\frac{1}{2}r_2^{-1}(t)d'(t)\left[R(t)m_x(t) - Q(t)\right], \\ &\dot{m}_x = \left[A(t) - B(t)R(t)\right]m_x + B(t)m_\zeta(t), \quad m_x(t_0) = a, \end{aligned} \tag{5.20}$$

(an analogous result is obtained in [80] for systems with a white–noise excitation for $m_\zeta = 0$).

Thus, the control $u(t)$ controls not the trajectory $x(t)$ but the expected value $m_x(t)$.

5.1.2
Program Control for Systems with Wide–Band Disturbances

In the above considerations, specific features of stochastic systems were inconsiderable: the equations of the stochastic maximum principle (as well as a proof methods) are entirely adequate to respective relations in the deterministic case. In particular, the solution of the problem (5.12), (5.13) needs only a calculation of an expected value of the random excitation. But already in a slightly more general situation, when the system is still linear, but the components of the matrix $A(t)$ are random processes, an analytical determination of the expected value m_p and of the control (5.11) is a hard-solvable problem which need the use of approximate methods.

The first of the possible methods is an approximate solution of the equations of the maximum principle. Such an approach is used, for instance, in [64, 80, 104, 170] for an optimal control construction for systems with a small excitation of the white–noise form. The control is searched in a form of the small–parameter expansion; a solution of the deterministic problem is used as a generating solution.

The second way is more suitable for an analysis of systems with a weak control, studied in Chapter 3. In this case the system motion is close to free oscillations, and excitations and the control are of the same order of magnitude. Here the generating system is uncontrollable, i.e., a solution of the generating problem does not determine the control.

A decomposition approach, analogous to a partial averaging, is an effective method for solution of control problems of systems with a weak control.

Let the motion equations be reduced to the standard form

$$\dot{x} = f(t, x, u, \xi(t), \varepsilon) \tag{5.21}$$

by means of known transformations, ε being a small parameter. Considering the control and excitations to be of the same order of smallness, let reduce (5.21) to the form

$$\frac{dx_\varepsilon}{d\tau} = \varepsilon^{-1} f_1(\tau/\varepsilon^2, x_\varepsilon, \xi_\varepsilon) + \left[g(\tau/\varepsilon^2, x_\varepsilon, u) + f_2(\tau/\varepsilon^2, x_\varepsilon)\right], \quad x_\varepsilon(0) = a, \tag{5.22}$$

where g is a deterministic function, $Mf_1(t,\cdot,\xi) = 0$, $x_\varepsilon = x(\tau/\varepsilon^2)$, $\xi_\varepsilon = \xi(\tau/\varepsilon^2)$.

As was shown in Section 4.2, the effect of random excitations in the systems of such type becomes apparent for a time interval $\propto \varepsilon^{-2}$, thus a deterministic control in the function g has an order of magnitude 1.

It is considered that the solution of the system (5.22) weakly converges to some diffusion process for all admissible controls. In this case the solution process is divided into two steps: a disturbed system is replaced by the limit diffusion equation, and the optimal control is constructed on trajectories of the limit system.

Let us now substantiate the proposed procedure [77].

Let the following conditions be satisfied:

1) functions f_1, f_2 satisfy the conditions of smoothness, growth and mixing of Theorem A.14;

2) a function $g(t,x,u(t))$ satisfies the conditions of Theorem A.14 for all admissible controls $u(t)$;

3) limits

$$b_1(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} dt \int_t^T M f_{1x}(t,x,\xi(t)) f_1(s,x,\xi(s)) ds, \tag{5.23}$$

$$a_{jk}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} dt \int_t^T M f_1^{\,j}(t,x,\xi(t)) f_1^{\,k}(s,x,\xi(s)) ds, \tag{5.24}$$

$$A = \{a_{jk}\}, \qquad j,k = 1,\ldots,n,$$

$$b_2(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} f_2(t,x) dx, \tag{5.25}$$

exist in the domain determined by Theorem A.14.

Theorem 5.1. *Let conditions* 1) – 3) *be satisfied. Then for all admissible controls* $u(t)$ *the solution* $x_\varepsilon(\tau)$ *of* Eq. (5.22) *weakly converges to the solution* $x_{0\varepsilon}(\tau)$ *of the stochastic differential equation*

$$dx_{0\varepsilon} = \left[b(x_{0\varepsilon}) + g(\tau/\varepsilon^2, x_{0\varepsilon}, u)\right] d\tau + \sigma(x_{0\varepsilon}) dw, \tag{5.26}$$
$$x_{0\varepsilon}(0) = a$$

for $\varepsilon \to 0$ *for any finite interval* $0 \le \tau \le \tau_f$, $\tau_f = \varepsilon^2 t_f$.

Here $b = b_1 + b_2$, $\sigma\sigma' = A$.

This conclusion follows directly from Comment 3 to Theorem A.14.

Let us show that the disturbed system (5.22) can be replaced by the limit stochastic system (5.26) in the solution of an optimal control problem.

Let us find the program control $u_*(t)$ minimizing the functional

$$\Phi_\varepsilon(u) = MF\left(x_\varepsilon(\tau_f)\right) \tag{5.27}$$

on trajectories of the system (5.22) under the condition $u \in U$, i.e.

$$u_* = \arg\min \Phi_\varepsilon(u) | u \in U . \tag{5.28}$$

Let U_ε be a set of admissible controls of the system (5.22), $U_{0\varepsilon}$ is a set of admissible controls of the system (5.26). Let, further, the control $u_{0\varepsilon}$ minimize the functional $\Phi_{0\varepsilon}(u)$ on trajectories of the system (5.26)

$$u_{0\varepsilon} = \arg\min \Phi_{0\varepsilon}(u)|u \in U\,, \qquad \Phi_{0\varepsilon}(u) = MF\left(x_{0\varepsilon}\left(\tau_f\right)\right). \tag{5.29}$$

Theorem 5.2. *Let the conditions of* Theorem 5.1 *hold, and the controls* u_*, $u_{0\varepsilon}$, *determined by relations (5.22), (5.26), exist and belong simultaneously to domains of admissible controls of both systems. Then the estimate*

$$0 \le \Phi_\varepsilon(u_{0\varepsilon}) - \Phi_\varepsilon(u_*) \le C\varepsilon\,, \qquad \varepsilon \to 0 \tag{5.30}$$

holds true.

Here and below C are constants, independent of ε.

The proof of Theorem 1.2 following from the weak convergence of the solution of the system (5.22) to the diffusion process (5.26) is given in Section A.2.

Let us formulate optimality conditions for a program control in the system (5.26).

There exist various formulations of the stochastic maximum principle in minimization problems for functionals with constraints in the form of Itô's stochastic differential equations [144, 150, 169–171]. The linkage between different maximum conditions is discussed, for instance, in [195, 200].

The form of stochastic maximum principle suggested in [173, 195] is the most suitable for applications. In [195] the approach of McShane [186] to stochastic integration is given; the stochastic maximum principle being then a limit case.

Let a system dynamics be described by the equation

$$dx = b(t,x,u)dt + \sigma(t,x)dw\,, \qquad x(t_0) = a \in R_n\,, \tag{5.31}$$

a control $u(t) = U \in R_m$ is determined by the minimum condition of the functional

$$\Phi(u) = M\left[F\left(x\left(t_f\right)\right) + \int_0^{t_f} \varphi(t,x,u)dt\right] \tag{5.32}$$

on trajectories of the system (5.31).

Suppose that the functions b, σ, φ, F are continuous and continuously differentiable, and for all $x(t) \in R_n$, $t \in \left[t_0, t_f\right]$ satisfy conditions

1) $|\sigma(t,x)| \le C(1+|x|)$, $|\sigma(t,x) - \sigma(t,y)| \le C|x-y|$;
2) $|b(t,x,u) - b(t,y,v)| \le C(|x-y| + |u-v|)$, $|b(t,x,u)| \le C(1+|x|+|u|)$;
3) $|\varphi(t,x,u)| \le C\left(1+|x|^2+|u|^2\right)$, $|\varphi_x(t,x,u)| \le C\left(1+|x|+|u|^2\right)$;
4) $|F_x(x)| \le C(1+|x|)$

for all admissible controls.

Let us introduce an adjoint system

$$dp = -b_x'(t,x,u)pdt - \sum_{k=1}^{n} \sigma_x^k(t,x)pdw_k + \varphi_x(t,x,u)dt\,, \tag{5.33}$$

where σ^k is the k-th column of the matrix σ, so

$$\left(\sigma_x^k\right)_{ij} = \partial\sigma_{ik}/\partial x_j\,, \tag{5.34}$$

w_k is a one–dimensional Wiener process, and determine the function

$$H = -\varphi(t,x,u) + \left(p, b(t,x,u)\right). \tag{5.35}$$

Theorem 5.3 [173]. *Let the assumptions* 1) – 4) *hold and* $u \in U$, x *be a trajectory* [*solution of* Eq. (5.31)], p *be a respective Lagrange multiplier* [*solution of* Eq. (5.33)]. *If the control* u_* *minimizes the functional* $\Phi(u)$ *on trajectories of the system* (5.31), x_* *is a respective trajectory, i.e.,* $\Phi(u_*) \le \Phi(u)$, *then*

$$u_* = \arg\max_{u \in U} MH(t,x,u,p)\,. \tag{5.36}$$

holds true.

Theorem 5.3 can be treated as a particular case of the stochastic maximum principle (Section A.5) for systems described by Eqs. (5.30). A detailed deduction of Theorem 5.3 is in [177], some examples are also given there.

It follows form (5.35), (5.36) together with (5.11) that an optimal program control $u(t)$ depends on the expected value $Mp(t)$. This problem can be easily solved analytically if the system (5.31) is linear, and the quality criterion is linear or quadratic. At the same time, a solution of equations of the stochastic maximum principle for the initial disturbed system is linked with considerable hardships even for a linear case. Thus, a replacement of the disturbed system with the limit one simplifies a computational procedure.

Note that conditions 1) and 2) of Theorem imposes constraints to continuity and the growth character of coefficients of the limit system. Below we will analyze examples, for which these conditions are satisfied.

5.1.3 Periodic Control of Parametric Disturbances of Linear Systems

The system dynamics is described by the equation

$$\ddot{z} + \left[A + \varepsilon\Xi(t)\right]z + 2\varepsilon^2\beta\dot{z} = \varepsilon^2 Gu\,, \tag{5.37}$$
$$z(t_0) = a_1\,, \quad \dot{z}(t_0) = a_2\,.$$

Here $z \in R_n$, $u \in R_m$, $A = \Lambda^2$, $\Lambda = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\}$, $\Xi(t) = \{\xi_{ij}(t)\}$ is a matrix of random disturbances, β is a matrix of dissipation coefficients (without the loss of generality it can be considered that $\beta = \mathrm{diag}\{\beta_{jj}\}$), $G = \{g_{ij}\}$ is a constant matrix of amplification coefficients of respective dimension. Components $\xi_{ij}(t)$ of the matrix $\Xi(t)$ are supposed to be stationary and stationary connected random processes satisfying the conditions of Theorem A.14.

If there are no random disturbances, then free oscillations of the system die on the interval $t_f = O(\varepsilon^{-2})$. In Section 4.2 it is shown that random disturbances can lead to instability if spectral densities $S^{\xi}_{jk}(\lambda_j \pm \lambda_k)$ of disturbances $\xi_{ij}(t)$ are sufficiently large (the stochastic analog of the main parametric resonance). In this case the aim of the control is to counteract the oscillation development. Obviously, the program control can not stabilize the system. At the same time, if the mean work of the guiding excitation is negative, i.e., the control has a direction opposite to the movement, then the oscillation intensity can be diminished.

There exist many criteria characterizing dynamic properties of the system. If R_j is an oscillation amplitude for the j-th main coordinate, then the value

$$MR_j^2(t) = M\left(z_j^2(t) - \lambda_j^{-2}\dot{z}_j^2(t)\right) \tag{5.38}$$

characterizes a measure of the deviation of the system from the zero state at the moment t. The integral

$$M\int_0^{t_f} R'(t) r_1 R(t)dt \,,$$

where r_1 is a non-negative matrix, is a measure of the total deviation of the system from the rest on the interval $[0, t_f]$.

If it is necessary to provide a maximal closeness of the terminal state $z(t_f)$ to a zero one then a term $MR(t_f)r_3R'(t_f)$ is added to the integral criterion.

In order to account for the boundedness of the input variable, let us introduce the following quality criterion

$$\Phi_\varepsilon(u) = M\left\{\varepsilon^2\int_0^{t_f}\left[R'(t)r_1R(t) + u'(t)r_2u(t)\right]dt + R'(t_f)r_3R(t_f)\right\}, \tag{5.39}$$

where $r_2 > 0$. Below it is considered that r_1, r_2, r_3 are diagonal matrices of respective dimensions.

An introduction of the small parameter ε^2 before the integral means that the functional is finite, $O(1)$, over the intervals $t_f \propto \varepsilon^{-2}$ (comp. Section 3.1).

Let us construct the program control $u(t)$ minimizing the criterion (5.39) on trajectories of the system (5.37). Following the procedure of Section 1.2, let us reduce (5.37) to the standard form. In order to keep the linearity of the system, let us use the replacement of variables (4.59):

$$z = F_c(t)x^1 + F_s(t)x^2, \tag{5.40}$$
$$\dot{z} = -\Lambda F_s(t)x^1 + \Lambda F_c(t)x^2,$$
$$F_c(t) = \mathrm{diag}\{\cos\lambda_j t\},$$
$$F_s(t) = \mathrm{diag}\{\sin\lambda_j t\}, \qquad j = 1, \ldots, n,$$

and reduce (5.37) to the form

$$\dot{x}^k = \varepsilon\sum_{i=1}^{2}\left[\mu^{ki}(t) + \varepsilon\nu^{ki}(t)\right]x^i + \varepsilon^2 d^k(t)u(t), \qquad k = 1, 2, \tag{5.41}$$
$$x^1(0) = a_1, \qquad x^2(0) = \Lambda^{-1}a_2,$$

where components of matrices of coefficients μ^{ki}, ν^{ki} are determined by Eqs. (4.60) – (4.62), and

$$d^1(t) = -\Lambda F_s(t)G, \qquad d^2(t) = -\Lambda F_c(t)G. \tag{5.42}$$

Accounting for (5.38), re-write (5.39) in the form

$$\Phi_\varepsilon(u) = M\Biggl\{\left[(x^1)' r_3 x^1 + (x^2)' r_3 x^2\right]_{t=t_f} +$$
$$+\varepsilon^2\int_0^{t_f}\left[(x^1)' r_1 x^1 + (x^2)' r_1 x^2 + u' r_2 u\right]dt\Biggr\}. \tag{5.43}$$

Obviously, by means of the replacement

$$\dot{z}_{n+1} = \varepsilon^2\left(R' r_1 R + u' r_2 u\right), \qquad z_{n+1}(0) = 0,$$

$$\Phi_\varepsilon(u) = M\left[R'(t_f)r_3 R(t_f) + z_{n+1}(t_f)\right],$$

where the function R is determined by Eq. (5.38), the Bolza problem can be reduced to the Mayer problem with functionals of the type (5.27). Hence, Theorem 5.2 can be applied to the system (5.41), (5.43).

With a transition form (5.41) to the limit diffusion system, owing to (4.63), we get

$$dx_{0\varepsilon}^k = \left[\sum_{i=1}^{2} b^{ki}x_{0\varepsilon}^i - \beta x_{0\varepsilon}^k + d^k(\tau/\varepsilon^2)u\right]d\tau + \sigma^{kk}x_{0\varepsilon}^k dw^k, \qquad k = 1, 2, \tag{5.44}$$
$$x_{0\varepsilon}^1 = a_1, \qquad x_{0\varepsilon}^2 = \Lambda^{-1}a_2;$$

components of the matrices b^{ki} are expressed by Eqs. (4.69)

$$b^{11} = b^{22} = \mathrm{diag}\{\gamma_j\}, \quad b^{21} = -b^{12} = \mathrm{diag}\{\kappa_j\}, \quad j = 1, \dots, n. \tag{5.45}$$

The quasi-optimal control $u_{0\varepsilon}(\tau)$ minimizing the functional (5.43) on trajectories of the system (5.44), can be found from the condition

$$\frac{\partial}{\partial u} MH = 0,$$

where, owing to (5.35), (5.43), (5.44),

$$H = -\sum_{i=1}^{2} \left(x^k_{0\varepsilon}\right)' r_1 x^k_{02} - u' r_2 u + \sum_{i,k=1}^{2} \left(p^k\right)' \left[b^{ki} x^i_{0\varepsilon} - \beta x^k_{0\varepsilon} + d^k\left(\tau/\varepsilon^2\right) u\right], \tag{5.46}$$

i.e.,

$$u_{0\varepsilon}(\tau) = \frac{1}{2} r_2^{-1} \sum_{i=1}^{2} \left[d^k\left(\tau/\varepsilon^2\right)\right]' M p^k(\tau). \tag{5.47}$$

Here p^k are solutions to the adjoint system, corresponding to (5.44); owing to (5.32), (5.44)

$$\begin{aligned} dp^1 &= \left[-\left(b^{11} p^1 - \beta p^1 + b^{21} p^2\right) + 2 r_1 x^1_{0\varepsilon}\right] d\tau - \sigma^{11} p^1 dw^1, \\ dp^2 &= \left[-\left(b^{12} p^1 - \beta p^2 + b^{22} p^2\right) + 2 r_1 x^2_{0\varepsilon}\right] d\tau - \sigma^{22} p^2 dw^2, \end{aligned} \tag{5.48}$$

$$p^k\left(\tau_f\right) = -2 r_3 x^k_{0\varepsilon}\left(\tau_f\right), \quad \tau_f = \varepsilon^2 t_f, \quad k = 1, 2.$$

Thus, the problem is reduced to the calculation of expected values of the system (5.48); since the systems (5.44) and (5.48) are connected, it is necessary to examine the system of 4n equations. Denote

$$q^k_{0\varepsilon}(\tau) = M x^k_{0\varepsilon}(\tau), \quad m^k_{0\varepsilon}(\tau) = M p^k, \quad k = 1, 2.$$

Owing to the linearity of the system (5.48), it is possible to obtain the closed system for determination of the moments $q_{0\varepsilon}$, $m_{0\varepsilon}$

$$\begin{aligned} \frac{dq^k_{0\varepsilon}}{d\tau} &= \sum_{i=1}^{2} b^{ki} q^i_{0\varepsilon} - \beta q^k_{0\varepsilon} + \sum_{i=1}^{2} D^{ki} m^i_{0\varepsilon}, \\ \frac{dm^k_{0\varepsilon}}{d\tau} &= -\sum_{i=1}^{2} b^{ik} m^i_{0\varepsilon} + \beta m^k_{0\varepsilon} + 2 r_1 q^k_{0\varepsilon}, \end{aligned} \tag{5.49}$$

$$q^1_{0\varepsilon}(0) = a_1, \quad q^2_{0\varepsilon}(0) = \Lambda^{-1} a_2, \quad m^k_{0\varepsilon}\left(\tau_f\right) = -2 r_3 q^k_{0\varepsilon}\left(\tau_f\right), \tag{5.50}$$

$$D^{ki}\left(\tau/\varepsilon^2\right) = \frac{1}{2} d^k\left(\tau/\varepsilon^2\right) r_2^{-1} d^i\left(\tau/\varepsilon^2\right). \tag{5.51}$$

The solution of the system (5.49) can be simplified. Suppose that there exist limits

$$D_0^{ki} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} D^{ki}(t)dt\,, \tag{5.52}$$

continuously with respect to $t_0 \geq 0$.

Then it follows from Theorem A.5 that a solution of the non-stationary system (5.48) can be approximated by the solution of the system with constant coefficients

$$\begin{aligned} \frac{dq_{01}^k}{d\tau} &= \sum_{i=1}^{2} b^{ki} q_0^i - \beta q_0^k + D_0 m_0^i\,, \\ \frac{dm_0^k}{d\tau} &= -\sum_{i=1}^{2} b^{ik} m_0^i + \beta m_0^k + 2r_1 q_0^k \end{aligned} \tag{5.53}$$

with boundary conditions analogous to (5.50). Here, owing to (5.42), (5.51), (5.52)

$$D_0^{12} = D_0^{21} = 0, \qquad D_0 = D_0^{kk} = \operatorname{diag}\{\delta_1,\dots,\delta_n\}, \qquad k = 1, 2,$$

$$\delta_p = \frac{1}{2\lambda_p^2} \sum_{l=1}^{m} \left(g_{pl}\right)^2 r_{2l}^{-1}, \qquad p = 1, \dots, n.$$

Here the estimate

$$\left|m_{0\varepsilon}^k(\tau) - m_0^k(\tau)\right| \leq C_1\varepsilon^2,$$

holds true for all $0 \leq \tau \leq \tau_f$. Hence, replacing the control (5.47) with the expression

$$u_0(\tau,\varepsilon) = \frac{1}{2} r_2^{-1} \sum_{k=1}^{2} d^k\left(\tau/\varepsilon^2\right) m_0^k(\tau)\,, \qquad k = 1, 2, \tag{5.54}$$

we get

$$\left|u_{0\varepsilon}(\tau) - u_0(\tau)\right| \leq C_2\varepsilon^2\,. \tag{5.55}$$

Using the last estimate and the statement of Theorem 5.2, it is easy to show that the control $u_0(\tau,\varepsilon)$ is also quasi-optimal with respect to the initial system.

Thus, Eq. (5.54) together with Eqs. (5.53) determines the quasi-optimal control. In particular, for the system with one degree of freedom

$$\begin{aligned} &\ddot{z} + \left[\lambda^2 + \varepsilon\xi(t)\right] z + 2\varepsilon^2\beta\dot{z} = \varepsilon^2 u\,, \\ &z(0) = a_1\,, \qquad \dot{z}(0) = a_2\,, \end{aligned} \tag{5.56}$$

and the functional

$$\Phi_\varepsilon(u) = M\varepsilon^2 \int_0^{t_f} \left[z^2 + \lambda^{-2}\dot{z}^2 + ru^2\right] dt \tag{5.57}$$

we get the following expression for the quasi-optimal control:

$$u_0(\tau,\varepsilon) = \frac{1}{2r\lambda}\left[-m_0^1(\tau)\sin\lambda t + m_0^2(\tau)\cos\lambda t\right], \qquad \tau = \varepsilon^2 t\,, \tag{5.58}$$

where parameters m_0^1, m_0^2 satisfy the system of equations

$$\begin{aligned}
dm_0^1/d\tau &= -(\gamma-\beta)m_0^1 + \kappa m_0^2 + 2q_0^1\,,\\
dm_0^2/d\tau &= -(\gamma-\beta)m_0^2 - \kappa m_0^1 + 2q_0^2\,,\\
dq_0^1/d\tau &= (\gamma-\beta)q_0^1 - \kappa q_0^2 + m_0^1/2r\lambda^2\,,\\
dq_0^2/d\tau &= (\gamma-\beta)q_0^2 + \kappa q_0^1 + m_0^2/2r\lambda^2\,,
\end{aligned} \tag{5.59}$$

with boundary conditions

$$q_0^1(0) = a_1\,, \qquad q_0^2(0) = \lambda^{-1}a_2\,, \qquad m_0^1(0) = m_0^2(0) = 0\,. \tag{5.60}$$

Coefficients γ, κ are determined by Eq. (5.45).

In order to simplify computations, let us suppose that the mixing time of the process $\xi(t)$ is sufficiently smaller than the period of free oscillations of the system. Then it is possible to consider $\kappa \approx 0$ [see comments to Eqs. (4.70) – (4.72)], and the system can be divided into two independent sub-systems

$$\begin{aligned}
dq_0^k/d\tau &= (\gamma-\beta)q_0^k + m_0^k/2r\lambda^2\,,\\
dm_0^k/d\tau &= -(\gamma-\beta)m_0^k + 2q_0^k
\end{aligned} \tag{5.61}$$

with respective initial conditions. From (5.60), (5.61) we have

$$m^k(q) = C_k \sinh\nu(\tau - \tau_f)\,, \tag{5.62}$$

where

$$\nu^2 = (\gamma-\beta)^2 + \left(r\lambda^2\right)^{-1}, \tag{5.63}$$

$$C_1 = 2a_1\left[\nu\cosh\tau_f - (\gamma-\beta)\right], \qquad C_2 = 2a_2\lambda^{-1}\left[\nu\cosh\tau_f - (\gamma-\beta)\right].$$

Substituting (5.62), (5.63) into (5.58), we get

$$u_0(\tau,\varepsilon) = r^{-1}\lambda^{-2}\left[\nu\cosh\tau_f - (\gamma-\beta)\right]\sinh\nu(\tau-\tau_f)\left[-a_1\sin\lambda t + a_2\lambda^{-1}\cos\lambda t\right],$$
$$\tau = \varepsilon^2 t\,. \tag{5.64}$$

5.2 The Method of Dynamic Programming for Optimal Control Synthesis for Disturbed Systems

In Section 5.1 it was shown that in the absence of information about the behavior of the system at each moment of time t, a control can be formed as a function of time. A program control and feedback control for deterministic systems lead to the same result: the control can not provide the smaller value of the criterion than under the program control. It is linked with the following. The program control depends on the initial state of the system at moment t_0 and, knowing this initial state and the control, being used up to the moment t, the current state of the system at the moment t can be uniquely determined. Thus, an observation of the current state of a deterministic system does not provide any new information in comparison with the knowledge of the initial state. As a matter of fact, owing to the solution uniqueness for a given initial condition, the input control excitation is the same at each moment of time, the program control is being chosen before, and the control synthesis is successively realized.

Two sufficiently different approaches – the control synthesis and the program control – are entirely equivalent in the case of deterministic systems.

In an optimization problem for a stochastic system, these methods being principally different from the physical point of view result in different variants of the control problem. The program control, as a matter of fact, controls only a mean value of an unknown solution (it is especially obvious in an example of a linear system), while the feedback control directly controls the values of the state vector and gives a lower value of the quality criterion. A strict linkage of solutions of these two problems for guided diffusion processes is shown, for instance, in [200].

A problem of the optimal control synthesis is analyzed below on a basis of the dynamic programming method.

The method of dynamic programming suggested by R. Bellman provides a method of the control synthesis for Markov processes, i.e., for a case when the future of the system is entirely determined by current values of phase variables.

In Section 4.1. it was shown that continuous Markov processes are described by stochastic differential equations of the form

$$dx = b(t,x)dt + \sigma(t,x)dw\,, \qquad x(t_0) = a\,, \tag{5.65}$$

where $x(t)$ is a n–dimensional vector of phase variables, w is a standard l–dimensional Wiener process, b, σ are a drift vector and a diffusion matrix of respective dimensions.

Let

$$b = b(t,x,u)\,, \qquad \sigma = \sigma(t,x,u)\,,$$

and control be realized only on the class of Markov controls, i.e., it depends only on the current state $x(t)$ of the system at the moment t

$$u = u(t,x). \tag{5.66}$$

Then

$$dx = b^u(t,x)dt + \sigma^u(t,x)dw, \quad x(t_0) = a,$$
$$b^u = b(t,x,u(t,x)), \quad \sigma^u = \sigma(t,x,u(t,x)), \tag{5.67}$$

and the Markov process $X^{t_0,a} = x(t)$ is realized in the guided system.

If the system dynamics is described by the equation

$$\dot{x} = f(t,x,\xi(t)) + g(t,x,u), \tag{5.68}$$

and process components $\xi(t)$ are different from the white noise, then the problem can be solved in terms of the dynamic programming only when a new expanded vector of variables $X(t)$ can be constructed, such that the phase vector $x(t)$ is entirely determined by the components of the expanded vector, and the process $X(t)$ is a Markov one. Such variables $X(t)$ were called 'sufficient coordinates' in [121].

In the simplest case when $\xi(t)$ is a stationary random process with fractional–rational spectral density, treated as a result of action of a linear filter on the white noise, the motion equation (5.68) is supplemented by the liner filter equation, and the dynamic programming equation is written for a new expanded system [8, 90, 136]. Some methods of construction of sufficient coordinates and of Bellman's equation for more complicated cases are given in [136].

It can be naturally supposed that the solution of an optimal problem should be close to the Bellman's equation of some limit system when the excitation converges (in some sense) to the process of the white–noise type. This fact is proved in [168, 176, 182] for excitations which are of the Markov–chains type and which weakly converge to a continuous diffusion process. It will be shown further that this result holds true for a class of weakly controlled systems with a wide-band excitation.

Section 5.2 consists of two parts. At the beginning, a formal construction of the dynamic programming equation for controlled Markov processes is given. Then the control problems for oscillatory systems with excitations different from the white noise are examined. The general approach is analogous to the one developed in Section 5.1: it is proved that for any admissible control the disturbed process converges to a diffusion one, and the optimal control on the trajectories of the limit diffusion system should be found.

5.2.1 Equations of Dynamic Programming

Let us give a formal construction of the equation of dynamic programming which determines the optimal control synthesis. Neither strict formulation for conditions under which the equation holds true nor its proof will be given. A strict mathematical statement of the dynamic programming method can be found, for instance, in [94, 127].

Let us choose once more the Bolza criterion (5.30) as a minimizing one. Since the system's behavior is entirely determined by its initial state, then the Bolza criterion can be written in the form

$$J(s,x,u) = M_{s,x}\left\{\int_s^{t_f} \varphi\big(t,x(t),u(t,x(t))\big)dt + F\big(x(t_f)\big)\right\}, \qquad (5.69)$$

where $M_{s,x}$ is a conditional expected value [see (4.13)]; $x(s)=x$. Then the functional of the type (5.32)

$$\Phi(u) = M\left\{\int_{t_0}^{t_f} \varphi\big(t,x(t),u(t,x(t))\big)dt + F\big(x(t_f)\big)\right\}, \qquad u \in U, \qquad (5.70)$$

determined on the trajectories of the system (5.66) for initial conditions $x(t_0)=a$ can be treated as

$$J(t_0,a,u) = \Phi(u). \qquad (5.71)$$

Let u_* be some optimal control, i.e.,

$$J(s,x,u_*) = \min_{u \in U} J(s,x,u), \qquad (5.72)$$

and for $u_* = u(s,x)$

$$J(s,x,u_*) = V(s,x). \qquad (5.73)$$

In order to obtain formally an equation for the function $V(s,x)$, let us fix the control u_* and write the differentiation relation (4.10) for the function $V(t,x(t))$. Then

$$V(s,x) = -M_{s,x}\int_s^t \left(V_z + \mathcal{L}^u V\right)dz + M_{s,x}V(t,x(t)) \qquad (5.74)$$

for $s < t \le t_f$. Here $V_z = \partial V(z,x)/\partial z$ and

$$\mathcal{L}^u = \left(b, \frac{\partial}{\partial x}\right) + \frac{1}{2} \mathrm{Tr}\, A^u \frac{\partial^2}{\partial x^2} \tag{5.75}$$

where $A^u = \sigma^u \left(\sigma^u\right)'$. The integral function is calculated in the point $(z, x(z))$, where $x(z) = X^{s,x}(z)$.

On the other hand, suppose that an observer uses some control u at $s \le z \le t$, and the optimal control u_* for $z > t$, i.e.,

$$u_1 = \begin{cases} u(z, x(z)), z \le t, \\ u_*(z, x(z)), z > t. \end{cases}$$

Then the value of the criterion $J(s, x, u_1)$ should not be smaller than $V(s, x)$. In other words, owing to (5.69), Eq. (5.76) can be written as

$$J(s,x,u) = M_{s,x} \int_s^t \varphi(z, x(z), u(z, x(z)))dz + M_{s,x} J(t, x(t), u_*),$$

and

$$V(s,x) \le J(s, x, u_1), \quad V(t, x(t)) = J(t, x(t), u_*), \tag{5.76}$$

i.e.,

$$V(s,x) \le M_{s,x} \int_s^t \varphi(z, x(z), u(z, x(z)))dz + M_{s,x} J(t, x(t), u_*). \tag{5.77}$$

An equality in (5.77) holds true if the optimal control $u = u_*$ is used over the interval $[s, t]$. Subtracting (5.74) from (5.77), dividing it with $t - s$ and taking the limit for $t \to s_+$, we get

$$V_s(s,x) + \mathcal{L}^u V(s,x) + \varphi(s,x,u) \ge 0, \tag{5.78}$$

and it is an equality when $u = u_*$. So, the dynamic programming equation

$$\min_{u \in U}\left[V_s(s,x) + \mathcal{L}^u V(s,x) + \varphi(s,x,u)\right] = 0, \tag{5.79}$$

is formally obtained for the function $V(s, x)$.

The boundary condition

$$V(t_f, x) = F(x) \tag{5.80}$$

follows directly from the definition of the function $V(s, x)$. Here $\mathcal{L}^u$ is a generating differentiation operator (5.75) of the controlled Markov process (5.66).

The minimum value of the optimality criterion (5.70) corresponds to the solution $V(s,x)$ of Eqs. (5.79), (5.80). Owing to (5.70) – (5.73),

$$V(t_0,a)=\min_{u\in U} M\int_{t_0}^{t_f}\varphi(t,x,u)dt+F\big(x(t_f)\big). \tag{5.81}$$

Let us give the form of Bellman's equation for some optimal control problems.

In the problem of an optimal high-speed action, the moment τ_u of reaching for the first time of the boundary Γ of some set Q by the trajectory of the system (5.66), i.e.,

$$\Phi(u)=M\tau_u \tag{5.82}$$

for the constraint $u\in U$ or, in more general case,

$$\Phi(u)=M\int_{t_0}^{t_u}\varphi(t,x,u)dt\,. \tag{5.83}$$

Suppose that $x(t_0)\in R_n\setminus Q$. Then the dynamic programming equation is analogous to (5.79)

$$\min_{u\in U}\big[V_s(s,x)+\mathcal{L}^u V(s,x)+\varphi(s,x,u)\big]=0 \tag{5.84}$$

for $x\in R_n\setminus Q$, and boundary conditions have the form

$$V(s,x)=0,\qquad x\in\Gamma \tag{5.85}$$

(if the process begins on the set Γ then the time of high-speed actions is obviously equal zero).

The strict substantiation of the optimality principle, of correctness of the Bellman's differential equation, of the solvability of the synthesis problem (i.e., the existence and uniqueness of the solution of Eq. (5.67) for a found optimal solution) wander off from the theme of this work. A correct mathematical theory of guided diffusion processes can be found, for instance, in [94, 127]. It is proved, in particular, that the optimal control is determined by the Eqs. (5.79), (5.80) [or (5.84), (5.85)] if the solution $V(s,x)$ exists, and the function $V(s,x)$ is two times continuously differentiable with respect to x and one time with respect to s within the domain and is continuous on its boundary. At the same time, the optimal control exists and is determined by the dynamic programming equation also in the cases when the function $V(s,x)$ is not sufficiently smooth. Below it is supposed that the solution of the synthesis problem exists and is determined by the solution of the dynamic programming equation.

An exact solution of the Bellman's equation can be obtained only for some particular cases. At the present time, a technique of the method of small disturbances is developed for the approximate solution of Bellman's equations with small diffusion coefficients, with weak non-linearities, etc. [90, 136, 145, 176]. But all these results concern only systems of the type (5.67) or systems, reducible to them.

Control problems with a wide-band excitation, which is different, generally speaking, from the white noise, are examined below. The problem of the optimal control existence for such systems is not discussed. If the disturbance is a component of the Markov process then this problem can be solved by means of the phase space expansion and by the 'supplement' of the motion equations up to the Itô's system of equations [90, 122]. If the disturbance is bounded with probability 1, then the existence problem can be solved in the same way as for a deterministic case.

Below it is always supposed that an optimal control in the disturbed system and in the system constructed by means of the diffusion approximation exists.

5.2.2 Optimal Control for Systems with Wide–Band Random Disturbances [79]

Let dynamics of an oscillatory system be described by the equation

$$\ddot{z} + Az + \varepsilon F_1\left(t, z, \dot{z}, \xi(t)\right) + \varepsilon^2 F_2\left(t, z, \dot{z}\right) = \varepsilon^2 G\left(t, z, \dot{z}, u\right), \tag{5.86}$$
$$z(0) = a_1, \qquad \dot{z}(0) = a_2,$$

where $z \in R_n$, $A = \operatorname{diag}\left\{\lambda_1^2, \ldots, \lambda_n^2\right\}$, $u \in U \subset R_m$, vectors F_1, F_2 reflect an effect of additional non-linear non-conservative and disturbing factors, $\xi \in R_l$ is a random process.

An introduction of a small parameter ε in the system (5.86) is caused by the same reasons as in deterministic systems.

It is supposed that the movement is close to free oscillations for bounded time intervals, and that the effect of disturbances and control is observed for time intervals $O\left(\varepsilon^{-2}\right)$. Such an approach is justified, in particular, if the random disturbances cause instability of the system and the control is aimed on the minimization of deviations from the equilibrium state.

Another problem can be also examined: a system moves in a weak force field, and weak controls form the movement for time intervals $O\left(\varepsilon^{-2}\right)$. This problem was studied in detail in [134] for a deterministic case; here the same model is examined but with an account for random disturbances.

Independent of the physical concretization, suppose that a moment of the process end t_f is fixed, and let us limit our considerations by the Mayer problem.

Let us reduce the system (5.86) to the standard form in order to use the diffusion approximation. For this purpose, introduce a replacement (4.39)

$$z_j = e^{y_j} \cos\left(\lambda_j t + \varphi_j\right), \quad \dot{z}_j = -\lambda_j e^{y_j} \sin\left(\lambda_j t + \varphi_j\right). \tag{5.87}$$

Then the system (5.76) gets the form

$$\dot{x} = \varepsilon f_1\left(t, x, \xi(t)\right) + \varepsilon^2 \left[g(t, x, u) + f_2(t, x)\right]. \tag{5.88}$$

Here $x = (y, \varphi)$ is a generalized vector of slow variables, functions f_1, f_2, g are obtained by respective transformations of the vectors F_1, F_2, G (see Section 4.1). It can be easily proved that the vector g would directly depend on the fast time t even for $G = G(u)$.

A measurable function $u(t, x)$ will be named *an admissible control* if there exists a unique solution of Eq. (5.88) for all $u = u(t, x)$ and $u \in U$.

It is supposed that the functions f_1, f_2 satisfy the conditions of smoothness, growth and mixing of Theorem A.14, and that the function $g(t, x, u(t, x))$ also satisfies conditions of Theorem A.14 for all admissible controls.

An approximated solution of the optimal control problem is constructed in the same way as in Section 5.1.

Introducing a new variable $\tau = \varepsilon^2 t$ and with $x_\varepsilon = x\left(\tau/\varepsilon^2\right)$, $\xi_\varepsilon = \xi\left(\tau/\varepsilon^2\right)$, rewrite (5.88) in the form

$$\frac{dx_\varepsilon}{d\tau} = \varepsilon^{-1} f_1\left(\tau/\varepsilon^2, x_\varepsilon, \xi_\varepsilon\right) + \left[g\left(\tau/\varepsilon^2, x_\varepsilon, u\right) + f_2\left(\tau/\varepsilon^2, x_\varepsilon\right)\right], \tag{5.89}$$
$$x_\varepsilon(0) = a.$$

Let $u = u(t, x)$ be some admissible control for which conditions implied on the function $g(t, x, u(t, x))$ are fulfilled. Then the solution $x_\varepsilon(t)$ of Eq. (5.89) weakly converges to the solution $x_{0\varepsilon}(t)$ of the Itô's equation

$$dx_{0\varepsilon} = \left[b\left(x_{0\varepsilon}\right) + g\left(\tau/\varepsilon^2, x_{0\varepsilon}, u\right)\right]d\tau + \sigma\left(x_{0\varepsilon}\right)dw, \quad x_{0\varepsilon}(0) = a \tag{5.90}$$

for $\varepsilon \to 0$, $0 \le \tau \le \tau_f$. Coefficients b, σ are calculated with the help of Eqs. (5.23) – (5-26).

Let us construct an approximately optimal control synthesis under the consideration that the quality criterion depends only on the slow variables. Let us at first study the Mayer problem with the functional $\Phi_\varepsilon(u) = MF\left(x\left(t_f\right)\right)$; $t_f = \varepsilon^{-2}\tau_f$, $\tau_f = O(1)$.

Let $u_*(t, x)$ be an optimal control minimizing the functional $MF\left(x_\varepsilon\left(t_f\right)\right)$ on trajectories of the initial system (5.89), $u_{0\varepsilon}(\tau, x)$ is a control minimizing the same functional $MF\left(x_{0\varepsilon}\left(t_f\right)\right)$ on trajectories of the system (5.90). Then, using the same way as in Section 5.1, it can be shown that

$$0 \le \Phi_\varepsilon\left(u_{0\varepsilon}\right) - \Phi_\varepsilon\left(u_*\right) \le C\varepsilon, \quad \varepsilon \to 0. \tag{5.91}$$

Here and below C, C_j, c are constant values independent of ε.

The dynamic programming equation determining the control $u_{0\varepsilon}$ has the form [see (5.79), (5.80)]

$$V_\xi^\varepsilon + \mathcal{L}_0 V^\varepsilon(\zeta,x) + H^u\left(\zeta/\varepsilon^2, x, V_x^\varepsilon\right) = 0, \quad V^\varepsilon\left(\tau_f, x\right) = F(x). \tag{5.92}$$

Here

$$\mathcal{L}^u = \sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{5.93}$$

$$H^u(t,x,q) = \min_{u \in U}\left(g(t,x,u),q\right). \tag{5.94}$$

Thus, the control

$$u_{0\varepsilon}(\zeta,x) = \arg H^u\left(\zeta/\varepsilon^2, x, V_x^\varepsilon(\zeta,x)\right) = U\left(\zeta/\varepsilon^2, x, V_x^\varepsilon(\zeta,x)\right) \tag{5.95}$$

is uniquely determined, and the functional

$$\left|\Phi_\varepsilon\left(u_{0\varepsilon}\right) - V^\varepsilon(0,a)\right| \le C_1\varepsilon. \tag{5.96}$$

A transition from the system (5.89) to the system (5.90) allowed the construction of additional equations (5.92) – (5.94) determining (5.95). Further simplifications are linked with the transformation of Eq. (5.92). In order to simplify it, let us use Theorem A.6 about the averaging of non-linear parabolic equations. Suppose that coefficients b_j, a_{ij}, H^u of Eq. (5.92) satisfy conditions of Theorem A.6. Let, further, $V^0(\zeta,x)$ be a solution of the averaged equation

$$V_\xi^\varepsilon + \mathcal{L}_0 V^0 + H_0\left(x, V_x^0\right) = 0, \quad V^0\left(\tau_f, x\right) = F(x). \tag{5.97}$$

where

$$H_0(x,q) = \lim_{T\to\infty}\frac{1}{T}\int_0^T H^u(t,x,q)dt.$$

Then for $x \in S_x$, $q \in S_q$, where S is a bounded domain in R_n and for sufficiently small ε

$$\left|V^\varepsilon(\zeta,x) - V^0(\zeta,x)\right| \le C_2\varepsilon^2, \quad 0 \le \zeta \le \tau_f. \tag{5.98}$$

The smoothness requirement for coefficients of the operator $\mathcal{L}_0$ named in conditions of Theorem A.6 are satisfied if conditions of Theorem A.14, providing a transition to the system (5.90), hold.

The condition (5.96) links values of V^{ε} and V^{0}, and serves for determination of the value of $\Phi_{\varepsilon}(u_{0\varepsilon})$ in the first approximation. Let us show that the control

$$u_0(\zeta, x, \varepsilon) = U\left(\zeta/\varepsilon^2, x, V_x^0(\zeta, x)\right) \tag{5.99}$$

is quasi-optimal with respect to the initial system, i.e.,

$$0 \le \Phi_{\varepsilon}(u_0) - \Phi_{\varepsilon}(u_*) \le c\varepsilon. \tag{5.100}$$

Here $U(t, x, q)$ is the same functional dependency, found from the condition (5.94), as in Eq. (5.95).

Let $\tilde{x}_{\varepsilon}(\tau)$ be a solution of Eq. (5.89) for $u = u_0(\tau, x, \varepsilon)$. If the control u_0 is admissible then the function $g(t, x, u_0)$ satisfies conditions of Theorem A.14; in particular, the limit

$$\begin{aligned} g^0(\tau, x) &= \lim_{\varepsilon \to 0} \int_0^{\tau_0} g\left(\zeta/\varepsilon^2, x, U\left(\zeta/\varepsilon^2, x, V_x^0(\tau, x)\right)\right) d\zeta \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^{T} g\left(t, x, U\left(t, x, V_x^0(\tau, x)\right)\right) dt \end{aligned} \tag{5.101}$$

$$\tau_0 > 0, \quad \tau \in \left[0, \tau_f\right], \quad x \in S_x \subset R_n,$$

exists uniformly with respect to τ_0, τ, x.

It follows from the previous considerations that the value of the functional $\Phi_{\varepsilon}(u_{0\varepsilon}) = MF\left(\tilde{x}_{\varepsilon}(\tau_f)\right)$ satisfies the condition

$$\left|\Phi_{\varepsilon}(u_0) - W(0, a)\right| \le C_3 \varepsilon, \tag{5.102}$$

where $W(\zeta, x)$ is a solution of the inverse Kolmogorov's equation

$$W_{\zeta} + \mathcal{L}_0 W + \left(g^0(\zeta, x), W_x\right) = 0, \quad W\left(\tau_f, x\right) = F(x). \tag{5.103}$$

Let us construct the equation for the function $v = V^0 - W$. It follows form (5.97), (5.101), (5.103) that

$$v_{\zeta} + \mathcal{L}_0 v + \left(g^0(\zeta, x), v_x\right) = 0, \quad v\left(\tau_f, x\right) = 0. \tag{5.104}$$

Obviously, $v(\zeta, x) = 0$, $W(\zeta, x) = V^0(\zeta, x)$, i.e.,

$$\left|\Phi_{\varepsilon}(u_0) - V^0(0, a)\right| \le C_3 \varepsilon. \tag{5.105}$$

In its turn, it follows from (5.96), (5.98), (5.105) that

$$\left|\Phi_\varepsilon(u_{0\varepsilon})-\Phi_\varepsilon(u_0)\right|\le\left|\Phi_\varepsilon(u_{0\varepsilon})-V^\varepsilon(0,a)\right|+\left|\Phi_\varepsilon(u_0)-V^0(0,a)\right|$$
$$+\left|V^\varepsilon(0,a)-V^0(0,a)\right|\le(C_1+C_2\varepsilon+C_3)\varepsilon\,. \tag{5.106}$$

Comparing (5.101) and (5.106), we get the inequality (5.100).

For the sake of brevity, we limited our considerations to the Mayer problem. More general cases are examined analogously. Let the functional of the problem have the form

$$\Phi_\varepsilon(u)=M\left[\int_0^{\tau_f}\varphi(\tau/\varepsilon^2,x_\varepsilon,u)d\tau+F(x_\varepsilon(\tau_f))\right]. \tag{5.107}$$

By means of introduction of an additional variable $x_{n+1}(\tau)$ satisfying the equation

$$\frac{dx_{n+1}}{d\tau}=\varphi(\tau/\varepsilon^2,x_\varepsilon(\tau),u)\,,\qquad x_{n+1}(0)=0\,,$$

the problem is reduced to the previously studied one. The optimal control is derived from Eqs (5.92) – (5.95), (5.97) for

$$H^u(t,x,q)=\min_{u\in U}\left[(g(t,x,u),q)+\varphi(t,x,u)\right]. \tag{5.108}$$

The form of the function H_0 changes respectively.

Thus, the following result holds true:

Theorem 5.4. *Let*

1) *coefficients of* Eq. (5.91) *and of the functional* (5.107) *satisfy the mentioned conditions of smoothness, growth and mixing;*

2) *optimal controls* $u_*(\zeta,x,\varepsilon)$ *and* $u_{0\varepsilon}(\zeta,x,\varepsilon)$ *minimizing the functional* (5.107) *on trajectories of the system* (5.89) *and* (5.90), *respectively, exist and belong simultaneously to the sets of admissible controls of both systems*;

3) *the control* $u_0(\zeta,x,\varepsilon)$, *determined by* Eq. (5.99), *exist and belong to the set of admissible controls of the system* (5.89).

Then the control $u_0(\zeta,x,\varepsilon)$ *is quasi-optimal with respect to the initial system and the estimate* (5.100) *holds true.*

Theorem 5.4 holds true also for other types of optimal control problems. In particular, it can be shown that in the problem of the optimal high-speed action with the functional (5.82) the quasi-optimal control is calculated according to Eq. (5.99), and the function V^0 satisfies the stationary equation

$$\mathcal{L}_0V^0(x)+H_0(x,V_x^0(x))+1=0\,,\qquad x\in R_n\setminus Q\,,$$
$$V^0(x)=0\,,\qquad x\in\Gamma\,,\qquad x(0)\in R_n\setminus Q\,. \tag{5.109}$$

5.2.3 Periodic Control for Systems with Disturbances

We will limit our considerations below to the case of a system, dynamics of which is described by Eq. (4.38)

$$\ddot{z} + \left[A + \varepsilon \Xi(t)\right] z + \varepsilon^2 Z(z, \dot{z}) = \varepsilon F(t) + \varepsilon^2 G u ,$$
$$z(0) = a_1 , \quad \dot{z}(0) = a_2 . \tag{5.110}$$

Here $z \in R_n$, $A = \Lambda^2$, $\Lambda = \operatorname{diag}\{\lambda_j\}$; $\Xi(t) = \{\xi_{ij}(t)\}$; $F(t) = \{\zeta_i(t)\}$ is a matrix (vector) of random disturbances, $i, j = 1, \ldots, n$, $Z(z, z)$ is a vector of additional non-linear and non-conservative forces, $u \in U \subset R_m$ is a vector of control excitations; G is a constant matrix of respective dimension. It is supposed that the function Z is sufficiently smooth, processes $\xi_{ij}(t)$, $\{\zeta_j\}$ satisfy condition (A) of Theorem A.14.

It is additionally supposed that eigenfrequencies of the system are not linked by resonance relations, i.e.,

$$\lambda_j / \lambda_k \neq m/r ; \quad k \neq j ; \quad j, k = 1, \ldots, n; \; m, r = 1, 2, \ldots$$

We will reduce the system to (5.110) by means of the replacement (5.87) to the standard form

$$\dot{y} = \varepsilon f^{11}(t, y, \varphi, \xi(t)) + \varepsilon^2 \left[f^{12}(t, y, \varphi) + g^1(t, y, \varphi, u)\right],$$
$$\dot{\varphi} = \varepsilon f^{21}(t, y, \varphi, \xi(t)) + \varepsilon^2 \left[f^{22}(t, y, \varphi) + g^2(t, y, \varphi, u)\right], \tag{5.111}$$

where $y = (y_1, \ldots, y_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, functions f^{ij} are expressed by Eqs. (4.41) (with respective change of variables),

$$g_j^1 = -\left(\lambda_j e^{y_i}\right)^{-1} \sin\left(\lambda_j t + \varphi_j\right) \sum_{k=1}^{m} g_{jk} u_k ,$$
$$g_j^1 = -\left(\lambda_j e^{y_i}\right)^{-1} \cos\left(\lambda_j t + \varphi_j\right) \sum_{k=1}^{m} g_{jk} u_k , \quad j = 1, \ldots, n. \tag{5.112}$$

In periodic control problems, the problem functional does not, as a rule, depend on the oscillation phase. Let us show that in this case a solution of the averaged equation of dynamic programming does not depend on φ.

Let us introduce a $2n$-dimensional vector $x = (y, \varphi)$. The limit system of equations (5.90) for variables $y_{0\varepsilon}$, $\varphi_{0\varepsilon}$ is then written in the form

$$dy_{0\varepsilon} = \left[b^1\left(y_{0\varepsilon}\right) + g^1\left(\tau/\varepsilon^2, y_{0\varepsilon}, \varphi_{0\varepsilon}, u\right)\right]d\tau + \sigma^{11}\left(y_{0\varepsilon}\right), \tag{5.113}$$
$$d\varphi_{0\varepsilon} = \left[b^2\left(y_{0\varepsilon}\right) + g^1\left(\tau/\varepsilon^2, y_{0\varepsilon}, \varphi_{0\varepsilon}, u\right)\right]d\tau + \sigma^{22}\left(y_{0\varepsilon}\right),$$

where coefficients b^j, σ^{ij} are calculated according to Eqs. (4.43) – (4.47) and does not depend on φ, $\tau = \varepsilon^2 t$.

Let the problem functional have the form

$$\Phi_\varepsilon(u) = M\left\{F\left(y\left(\tau_f\right)\right) + \int_0^{\tau_f}\left[\psi(y) + u'ru\right]d\tau\right\}. \tag{5.114}$$

where $r = \operatorname{diag}\left\{r_j\right\}$, $r_j > 0$ $j = 1$, ..., m; there are no constraints of the type $u \in U$. Then the operator $H^u(t, y, \varphi, q)$ can be written in the form

$$H^u = \min_u\left\{\sum_{k=1}^{m}\left[\sum_{j=1}^{m} H_{jk}u_k + r_k u_k^2\right]\right\},$$

where, accounting for Eq. (5.112),

$$H_{jk}(t, y, \varphi, q) = -g_{jk}\left(\lambda_j e^{y_j}\right)^{-1}\left[q_j^1 \sin\left(\lambda_j t + \varphi_j\right) + q_j^2 \cos\left(\lambda_j t + \varphi_j\right)\right],$$

$$q_j^1 = \frac{\partial V^0}{\partial y_j}, \quad q_j^2 = \frac{\partial V^0}{\partial \varphi_j}. \tag{5.115}$$

From the minimum conditions we have

$$u_k = \frac{1}{2r_k}\sum_{j=1}^{n}\left(\lambda_j e^{y_j}\right)^{-1} g_{jk}\left[q_j^1 \sin\left(\lambda_j t + \varphi_j\right) + q_j^2 \cos\left(\lambda_j t + \varphi_j\right)\right], \tag{5.116}$$

and

$$H^u(t, y, \varphi, q) = -\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\left(2r_k\lambda_j e^{y_j}\right)^{-1} g_{jk}g_{ik}\left[q_j^1 \sin\left(\lambda_j t + \varphi_j\right)\right.$$
$$\left. + q_j^2 \cos\left(\lambda_j t + \varphi_j\right)\right]\left[q_i^1 \sin\left(\lambda_i t + \varphi_i\right) + q_i^2 \cos\left(\lambda_i t + \varphi_i\right)\right].$$

Averaging the function $H^u(t, y, \varphi, q)$, we get

$$H_0(y, q) = -\sum_{j=1}^{n}\sum_{k=1}^{m}\left(g_{jk}\right)^2\left(4r_k\lambda_j e^{y_j}\right)^{-1}\left[\left(q_j^1\right)^2 + \left(q_j^2\right)^2\right].$$

Thus, the averaged equation of dynamic programming (5.97) gets the form (with account for notations $q^1 = \partial V^0/dy$, $q^2 = \partial V^0/d\varphi$)

$$\frac{\partial V^0}{\partial \tau} + \sum_{j=1}^{n}\left[b_j^1(y)\frac{\partial V^0}{\partial y_j} + b_j^2(y)\frac{\partial V^0}{\partial \varphi_j}\right] + \frac{1}{2}\sum_{i,j=1}^{n}\left[a_{ij}^{11}(y)\frac{\partial^2 V^0}{\partial y_i \partial y_j} + a_{ij}^{22}(y)\frac{\partial^2 V^0}{\partial \varphi_i \partial \varphi_j}\right]$$

$$-\frac{1}{8}\sum_{j=1}^{n}\sum_{k=1}^{m} g_{jk}^{2}\left(r_k \lambda_j e^{2y_j}\right)^{-1}\left[\left(\frac{\partial V^0}{\partial y_j}\right)^2+\left(\frac{\partial V^0}{\partial \varphi_j}\right)^2\right]+\psi(y)=0; \qquad (5.117)$$

the boundary condition has the form

$$V^0(\tau, y)=F(y). \qquad (5.118)$$

Here a_{ij}^{ll} are coefficients of matrices $A^{ll}=\sigma^{ll}\left(\sigma^{ll}\right)'$, $l=1,2$.

Coefficients of Eq. (5.117) and boundary conditions (5.118) does not depend on φ, i.e.,

$$q^2=\frac{\partial V^0}{\partial \varphi}=0,$$

and the dynamic programming equation gets the simplified form

$$\frac{\partial V^0}{\partial \tau}+\sum_{j=1}^{n} b_j^1(y)\frac{\partial V^0}{\partial y_j}+\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}^{11}(y)\frac{\partial^2 V^0}{\partial y_i \partial y_j}$$

$$-\frac{1}{8}\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{g_{jk}^{2}e^{-2y_j}}{\lambda_j^2 r_k}\left(\frac{\partial V^0}{\partial y_j}\right)^2+\psi(y)=0, \qquad (5.119)$$

$$V^0(\tau_f, y)=F(y).$$

The control $u_0=\left(u_0^1,\ldots,u_0^n\right)$ gets, in its turn, the form

$$u_0^k=\frac{1}{2r_k}\sum_{j=1}^{n}\frac{g_{jk}e^{-y_j}}{\lambda_j}\frac{\partial V^0}{\partial y_j}\sin\left(\lambda_j t+\varphi_j\right). \qquad (5.120)$$

If the system is linear, $Z(z,\dot{z})=2\beta\dot{z}$, where β is a positively determined matrix with elements β_{ij}, then coefficients b_j^1, a_{ij}^{11} are determined by Eqs. (4.49) – (4.50):

$$b_j^1=\nu_j-\beta_{jj}, \quad a_{ij}^{11}=\alpha_{ij},$$

$$a_{ij}^{11}=\alpha_{jj}+d_{jj}e^{-2y_j}+e^{-2y_j}\sum_{\substack{k=1\\k\neq j}}^{n} q_{jk}e^{2y_k},$$

where constant parameters ν, α, d, q depend on spectral densities of processes ξ_{ij}, ζ_j.

Suppose that the functional bi-linearly depends on the oscillation amplitude, i.e.,

$$\psi(y) = \left(e^{y}\right)' \rho e^{y}, \quad \Gamma(y) = \left(e^{y}\right)' \Gamma e^{y}, \tag{5.121}$$

where ρ and Γ are diagonal matrices with components ρ_i, Γ_i, respectively. Then the function $V^0(\tau, y)$ should have the form

$$V^0(\tau, y) = \left(e^{y}\right)' P(\tau) e^{y} + Q(\tau), \tag{5.122}$$

where $P(\tau)$ is a diagonal matrix, Q is a scalar value. Components P_i of the matrix P satisfy the system of Riccati equations

$$\dot{P}_l + 2(v_l - \beta_{ll} + \alpha_{ll})P_l + \sum_{\substack{j=1 \\ j\neq l}}^{n} P_j q_{jl} - \frac{1}{2} P_l^2 \sum_{j=1}^{n} \frac{g_{jl}^2}{\lambda_l^2 r_j} + \rho_l = 0, \quad P_l(\tau_f) = \Gamma_l; \tag{5.123}$$

the linear equation

$$\dot{Q} + \sum_{l=1}^{n} d_{ll} P_l = 0, \quad Q(\tau_f) = 0 \tag{5.124}$$

serves for the determination of $Q(\tau)$.

It follows from the obtained relations with account for Eq. (4.50) that the matrix P does not depend on the spectral densities of the external excitation; these characteristics of coefficients d_{ll} influence only the function Q.

From (5.120), (5.1229 we have

$$\frac{\partial V^0}{\partial y_j} = 2P_j e^{2y_j},$$

$$u_0^k = \frac{1}{r_k} \sum_{j=1}^{n} \lambda_j^{-1} g_{jk} P_j e^{y_j} \sin(\lambda_j t + \varphi_j). \tag{5.125}$$

Accounting for the replacement (5.87), we will present (5.125) in the form

$$u_0^k = -r_k^{-1} \sum_{j=1}^{n} \lambda_j^{-2} g_{jk} P_j \dot{z}_j, \quad u_0 = -r^{-1} G' A^{-1} P \dot{z}. \tag{5.126}$$

Thus, the optimal control has a form of a feedback with respect to velocity; feedback coefficients depend only on characteristics of parametric excitation, an external excitation influencing only the value of the functional.

For the system with one degree of freedom

$$u_0 = -p\dot{z}\left(r\lambda^2\right)^{-1}, \tag{5.127}$$

where a coefficient p satisfies the equation

$$\dot{p} + 2(v - \beta + \alpha)p - \left(2r\lambda^2\right)^{-1} p^2 + \rho = 0, \tag{5.128}$$

$$p(\tau_f) = \Gamma, \quad v = \alpha = S_\xi(2\lambda)/8\lambda^2 .$$

A solution of other periodic control problems can be simplified analogously.

Now we will analyze the control problem for a vibroimpact system, dynamics of which is described by the equation

$$\ddot{x} + \Omega^2(1 + \varepsilon\xi(t))x + \varepsilon^2(2\beta\dot{x} + ux + \Omega^2\Delta) = 0 \tag{5.129}$$

and by impact conditions

$$x = 0, \quad \dot{x}_+ = -(1 - \varepsilon r^2)\dot{x}_- . \tag{5.130}$$

By means of replacement of variables (4.94) – (4.96)

$$x = -e^y\chi(\psi), \quad \dot{x} = -2\Omega e^y \chi_\psi(\psi), \quad \psi = 2\Omega(t - \varphi), \tag{5.131}$$

$$\chi(\psi) = -(2\Omega)^{-1}\sin\psi/2, \quad 0 < \psi < 2\pi, \tag{5.132}$$

Eq. (5.129) is reduced to the standard form

$$\dot{y} = -4\varepsilon\left[\Omega^2\chi\chi_t\xi(t) + \varepsilon(2\beta\chi_t + u\chi - \Omega^2\Delta e^{-y})\chi_t\right] - 2r\Omega\varepsilon^2\delta^{2\pi}(\psi), \quad y(0) = a,$$

$$\dot{\varphi} = -4\varepsilon\left[\Omega^2\chi^2\xi(t) + \varepsilon(2\beta\chi_t + u\chi - \Omega^2\Delta e^{-y})\chi\right], \tag{5.133}$$

where $\chi = \chi[2\Omega(t-\varphi)]$, $\chi_t = \chi_t[2\Omega(t-\varphi)]$, impact points t_k are determined by the condition (4.98).

Let us construct the control u providing a maximum impulse in a fixed time under the condition $|u| \le U_0$. The functional of the problem can be presented in the form

$$\Phi_\varepsilon(u) = M\exp(y(t_f)). \tag{5.134}$$

We search the solution using the discussed scheme. At first, the dynamic programming equation is written for the partially averaged diffusion system

$$dy_{0\varepsilon} = \left[(-r\Omega/\pi - \beta + v) - 4u\zeta_1\right]d\tau + \sigma_1 dw,$$

$$d\varphi_{0\varepsilon} = \left[-4u\zeta_2 + \Omega^2\Delta k(y_{0\varepsilon})\right]d\tau + \sigma_2 dw . \tag{5.135}$$

Here, owing to (4.103) – (4.105), $\zeta_1 = \chi\chi_t$, $\zeta_2 = \chi^2$, coefficients v and σ_1 are determined by Eq. (4.106)

$$v = \sigma_1^2 = \frac{\Omega^2}{8}S_\xi(2\Omega); \tag{5.136}$$

coefficients k and σ_2 have the form

$$k(y)=e^{-y}k_1\,,\qquad k_1=\frac{1}{T}\int_0^T\chi(2\Omega t)dt=\frac{1}{\pi\Omega}\,, \tag{5.137}$$

$$\sigma_2^2=\frac{16\Omega^4}{T}\int_0^T ds\int_{-\infty}^{\infty}\xi_2[2\Omega(s-\varphi)]\xi_2[2\Omega(t-\varphi)]K_\xi(t-s)dt$$
$$=\frac{1}{4}\left[S_\xi(0)+\frac{1}{2}S_\xi(2\Omega)\right].$$

Here $S_\xi(\lambda)$ is a spectral density of the process $\xi(t)$.

Eqs. (5.95), (5.96), (5.99) determining the quasi-optimal control u_0 get the form

$$H=-4u(q_1\zeta_1+q_2\zeta_2)\,, \tag{5.138}$$

$$u_0=-U_0\operatorname{sgn}(q_1\zeta_1+q_2\zeta_2)\,, \tag{5.139}$$

$$H^u=4U_0(q_1\zeta_1+q_2\zeta_2)\,.$$

Here $q^1=\partial V^0/dy$, $q^2=\partial V^0/d\varphi$, $V^0(\tau,y,\varphi)$ is a solution of the averaged equation of dynamic programming. It can be shown (as in the previous example) that coefficients of the averaged equation (5.97) does not depend on φ, i.e. $q_2=0$. Then

$$H^u=4U_0\left|\frac{\partial V^0}{\partial y}\zeta_1\right|,\qquad H_0=\frac{U_0}{\pi\Omega}\left|\frac{\partial V^0}{\partial y}\right|, \tag{5.140}$$

where $V^0(\tau,y)$ is a solution of the equation

$$\frac{\partial V^0}{\partial\tau}+\left(v-\frac{r\Omega}{\pi}-\beta\right)\frac{\partial V^0}{\partial y}+\frac{1}{2}\sigma_1^2\frac{\partial^2V^0}{\partial y^2}+\frac{U_0}{\pi\Omega}\left|\frac{\partial V^0}{\partial y}\right|=0\,, \tag{5.141}$$

$$V^0(\tau_f,y)=e^y\,.$$

A solution of Eq. (5.141) has the form

$$V^0(\tau,y)=P(\tau)e^y\,; \tag{5.142}$$

a coefficient P satisfies the equation

$$\dot P+\left(-\frac{r\Omega}{\pi}-\beta+\frac{3}{2}v\right)P+\frac{U_0}{\pi\Omega}|P|=0\,,$$
$$\tag{5.143}$$
$$P(\tau_f)=1\,.$$

A solution of Eq. (5.143) has a constant sign, and, owing to boundary conditions,

$P > 0$. Hence,

$$P(\tau) = e^{\delta(\tau_f - \tau)}, \qquad \delta = \frac{3}{2}\nu + \frac{U_0}{\pi\Omega} - \beta - \frac{r\Omega}{\pi} \tag{5.144}$$

and $\partial V^0/dy > 0$, i.e., owing to (5.139),

$$u_0 = -U_0 \operatorname{sgn} \zeta_1 . \tag{5.145}$$

Accounting for the relation $\operatorname{sgn} \xi_1 = \operatorname{sgn} \chi\chi_t = \operatorname{sgn} x\dot{x}$, we get

$$u_0 = -U_0 \operatorname{sgn} x\dot{x} . \tag{5.146}$$

At last, the value of the functional is

$$V^0(0,a) = P(0)e^a = e^{\alpha + \delta\tau_f} . \tag{5.147}$$

5.3 Control for Stationary Motion under Random Disturbances

It was shown in the analysis of deterministic systems that control problems are of special interest since they are connected with a guidance of working regimes of machines. Analogous problems arise also in the control of stochastic systems. If random excitations are small and cause only negligible deviations of the working regime from a nominal one, then – as the first approximation – an optimal control is determined by a solution of the deterministic problem [90, 110, 136]. This is not true for weak-controlled systems, when controls and disturbances have the same order of magnitude.

This section deals with weak–control problems. The principal scheme is based on a differential approximation and is analogous to the case studied in Section 5.2; only main characteristic features of the approximate solutions are discussed.

5.3.1 Stationary Quality Criterion

Suppose that the system dynamics is described by the equation

$$\dot{x} = f(t, x, u(t,x))\xi(t), \qquad x \in R_n, \qquad u \in R_m, \qquad \xi \in R_l . \tag{5.148}$$

Let $\xi(t)$ be a stationary dynamic process, $f(t,\cdot,u(t,\cdot))$ be a periodic (quasi-periodic) function with respect to time. Then there exists under certain conditions a unique solution $\bar{x}(t)$ of Eq. (5.148) with corresponding periodic (quasi-periodic) probability distribution [138]. If this solution is stable then it can be treated as a motion of the system (5.148) at the moment t which begins at $t_0 \to -\infty$, i.e.

$\bar{x}(t) = X^{t_0,a}(t)$ for $a \in K \subset R_n$, $t_0 \to -\infty$.

Suppose that a set of admissible controls generating non-zero stationary solutions of Eq. (5.148) is not empty. Then a control $u(t, x(t))$ should be found which minimizes the functional

$$\Phi(u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t, \bar{x}(t), u)dt \,|\, u \in u \tag{5.149}$$

for the stationary solution $\bar{x}(t)$ of the system (5.148).

Optimality conditions for the stationary control of a linear system of Itô's equations with coefficients which do not depend on t and the quadratic quality criterion are formulated in [39, 199].

A construction of Bellman's equations for a stationary system of Itô's equations is given in [90, 175, 178]; the existence of solution for an optimal problem was proved in [181]. Let us construct a formal control of dynamic programming for the system (5.148) with the functional (5.149) without discussion of existence conditions for an optimal control.

Introduce a function $V(s,x)$ which is a solution of the formalized equation

$$\frac{\partial V}{\partial s} + \mathcal{L}V = \gamma^u - \varphi(s,x,u), \tag{5.150}$$

where $\partial/\partial s + \mathcal{L}$ is a generating operator of the system (5.148)

$$\frac{\partial V}{\partial s} + \mathcal{L}V = \lim_{h\to 0}\left[M_{s,x}V(s,h,\bar{x}(s+h)) - V(s,x)\right], \tag{5.151}$$

where γ^u is some constant value. From the definition of the generation operator we have

$$M_{s,x}V(s+T, \bar{x}(s+T)) - V(s,x) = M\int_s^{s+T}\left[\gamma^u - \varphi(t, \bar{x}(t), u(t, \bar{x}(t)))\right]dt,$$

and for $\bar{x}(t) = X^{s,x}(t)$, $\bar{x}(s) = x$,

$$\frac{1}{T}M_{s,x}\int_s^{s+T}\left[\varphi(t,\bar{x}(t),u(t,\bar{x}(t)))\right]dt = \gamma^u + \frac{1}{T}\left[V(s,x) - M_{s,x}V(s+T,\bar{x}(s+T))\right]. \tag{5.152}$$

Thus, if the function $V(s,x)$ satisfies the condition

$$\lim_{T\to\infty}\frac{1}{T}M_{s,x}V(s+T,\bar{x}(s+T)) = 0 \tag{5.153}$$

for $u = u(s,x)$, and this limit exists continuously with respect to s, x, then the parameter γ^u determines the value of the functional (5.150)

$$\gamma^u = \Phi(u), \tag{5.154}$$

and it follows from (5.152) that

$$M_{s,x} \int_s^{s+T} \left[\varphi(t, \bar{x}(t), u(t, \bar{x}(t)))\right] dt = \gamma^u T + V(s,x) + \psi(s,T,x) \tag{5.155}$$

uniformly with respect to s, x, and $T^{-1}\psi(s,x,T) \to 0$ for $T \to \infty$.

If $u = u_*(t,x)$ is an optimal control with a unique respective stationary solution $\bar{x}_*(t)$ then, owing to (5.154),

$$\gamma = \min_{u \in U} \gamma^u = \Phi(u_*). \tag{5.156}$$

It follows form (5.151), (5.156) that

$$\gamma = \min_{u \in U} \left[V_s + \mathcal{L}V + \varphi(s,x,u)\right]. \tag{5.157}$$

If a system dynamics is described by the Itô's equation

$$dx = b(t,x,u)dt + \sigma(t,x)dw, \qquad x \in R_n, \tag{5.158}$$

and coefficients b, σ satisfy necessary smoothness conditions, then

$$\mathcal{L} = \sum_{i=1}^{n} b_i(t,x,u)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(s,x)\frac{\partial^2}{\partial x_i \partial x_j},$$

and the control $u(s,x)$ minimizing the functional (5.150) is determined by the equation

$$V_s(s,x) + \min_{u \in U}\left[\mathcal{L}V(s,x) + \varphi(s,x,u)\right] = \gamma. \tag{5.159}$$

If coefficients b, σ, φ do not depend on s, then $V = V(x)$, $u = u(x)$.

In analogy with [39], let us give the solution to the problem of an optimal control synthesis in a stationary system

$$dx = (Ax + Bu)dt + \sigma dw + \sigma_1 x dw, \tag{5.160}$$

with a quadratic quality criterion

$$\Phi(u) = \lim_{T \to \infty} \frac{1}{T} \int_0^T M\left[x'N_1 x + uN_2 u\right] dt. \tag{5.161}$$

Here a vector of phase coordinates $x \in R_n$, a control $u \in R_m$, w_1 is a scalar Wiener process, w is a l–dimensional Wiener process, A, B, σ, σ_1 are constant matrices of dimensions $n \times n$, $n \times m$, $n \times l$, $n \times n$, respectively, N_1, N_2 are symmetric matrices of dimensions $n \times n$ and $n \times m$, respectively.

The control $u = u(x)$ is considered to be admissible if

1) the function $u(x)$ satisfies the Lipschitz condition

$$|u(x) - u(y)| \le L|x - y|;$$

2) for $u = u(x)$ there exists an independent of initial conditions stationary solution $x(t,u) = \bar{x}(t)$ of the system (5.160), such that

$$\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{T+t_0} M[\bar{x}(t)]^2 \, dt = \text{const} < \infty .$$

The Bellman's function of the problem under study is stationary $V = V(x)$, and the Bellman's equation (5.159) gets the form

$$\inf_u \left[V_x'(Ax + Bu) + x'N_1 x + u'N_2 u + \frac{1}{2}\left(x'\sigma_1 V_{xx}\sigma_1 x + \mathrm{Tr}\,\sigma V_{xx}\sigma\right)\right] = \gamma . \qquad (5.162)$$

It follows form (5.162) that the optimal control $u_*(x)$ is connected with the Bellman's function by the relation

$$u_*(x) = -\frac{1}{2} N_2^{-1} B' V_x(x),$$

and the function $V(x)$ has the form

$$V(x) = x'Px . \qquad (5.163)$$

Substituting (5.163) into (5.162), we get

$$\gamma = \mathrm{Tr}\,\sigma' P \sigma , \qquad (5.164)$$

where a symmetric matrix P is a solution of the equation

$$A'P + PA - PBN_2^{-1}B'P + N_1 + \sigma_1' P \sigma_1 = 0 . \qquad (5.165)$$

In analogy with [39], it can be shown that for sufficiently small $|\sigma_1|$ there exists the unique positively determined matrix P – the solution of Eq. (5.165). If the solution of Eq. (5.165) is found then the control

$$u_*(x) = -N_2^{-1} B' P x = -Kx \qquad (5.166)$$

is optimal, and there exists a unique stationary solution $\bar{x}(t)$ of the equation

$$dx = (A - BK)x + \sigma dw + \sigma_1 x dw .$$

Owing to (5.165), the matrix P of feedback coefficients depends only on σ_1, characterizing a parametric disturbance, and does not depend on σ; an external disturbance affects only the value of the functional (5.164).

5.3.2 Control for Stationary Motion in Systems with Wide–Band Random Disturbances

Let us examine once more a weak-controlled system. Omitting all initial considerations, consider that motion equations of the system are reduced to the standard form

$$\frac{dx}{dt} = \varepsilon f(t,x,\xi(t)) + \varepsilon^2 g(t,x,u), \tag{5.167}$$

$x \in R_n$, $u \in U \subset R_m$, $\xi \in R_l$ is a random disturbance, and construct the control $u_* = u(t,x(t))$, minimizing the functional (5.149) for the stationary motion $\bar{x}(t)$ of the system (5.167).

According to the general ideas of the diffusion approximation, we will introduce a slow variable $\tau = \varepsilon^2 t$ and re-write (5.167) in the form

$$\frac{dx_\varepsilon}{dt} = \frac{1}{\varepsilon} f\left(\frac{\tau}{\varepsilon^2}, x_\varepsilon, \xi_\varepsilon\right) + g\left(\frac{\tau}{\varepsilon^2}, x_\varepsilon, u\right), \tag{5.168}$$

where $x_\varepsilon = x(\tau/\varepsilon^2)$, $\xi_\varepsilon = \xi(\tau/\varepsilon^2)$. We will simplify the problem, replacing the disturbed system (5.168) with a more simple limit diffusion equation

$$dx_{0\varepsilon} = \left[b(x_{0\varepsilon}) + g(\tau/\varepsilon^2, x_{0\varepsilon}, u)\right]d\tau + \sigma(x_{0\varepsilon})dw, \tag{5.169}$$

and will build the control $u_{0\varepsilon}$, minimizing the functional (5.149) for the stationary solution $\bar{x}_{0\varepsilon}(\tau)$ of Eq. (5.169).

A replacement of the system (5.168) with a diffusion approximation is possible if coefficients f, g and the function $u(t,x)$ are such that the right-hand part of Eq. (5.168) satisfies conditions of Theorem 4.4. Let us remind that besides conditions of smoothness and mixing, the condition A.2 of Theorem 4.4, providing the solution stability, should be also satisfied. For guided systems this requirement can be formulated in the following way:

A. There exists two-times differentiable function $V = V(x)$, such that for sufficiently large $|x|$

$$\left[\mathcal{L}_0 + \left(g_0, \frac{\partial}{\partial x}\right)\right]V(x) \le -1. \tag{5.170}$$

Here

$$\mathcal{L}_0 = \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{5.171}$$

where $b = (b_1, \dots, b_n)$, $\sigma\sigma' = A = \{a_{ij}\}$, $i, j = 1, \dots, n$,

$$g_0(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} g(t, x, u(t,x))dt, \tag{5.172}$$

$u(t,x)$ is some control. It is supposed that the limit exists uniformly with respect to $t_0 > 0$, $x \in K \subset R_n$.

If condition A and conditions of Theorem 4.4 are satisfied for $u = u(\tau/\varepsilon^2, x)$, then there exists a stationary solution $\bar{x}_\varepsilon(\tau)$ of Eq. (5.169) which converges weakly for $\varepsilon \to 0$ to a stationary diffusion process with the generating operator

$$\mathcal{L} = \mathcal{L}_0 + \left(g_0, \frac{\partial}{\partial x}\right).$$

In its turn, the stationary solution $\bar{x}_{0\varepsilon}(\tau)$ of Eq. (5.169) for the same control also converges weakly for $\varepsilon \to 0$ to a stationary diffusion process with the same generating operator $\mathcal{L}$. Thus, the weak convergence of $\bar{x}_\varepsilon(\tau)$ to $\bar{x}_{0\varepsilon}(\tau)$ is provided.

It follows from above considerations, that the control $u(t,x) \in U$, generating a unique solution of Eq. (5.167), is related to the class of admissible solutions if the function $g(t, x, u(t,x))$ satisfies conditions of Theorem 4.4 and the condition (5.170) is fulfilled.

Thus, if the functions f, g satisfy conditions of Theorem 4.4 for all admissible controls, and there exist optimal controls u_* and $u_{0\varepsilon}$, minimizing the functional (5.149) on trajectories of the systems (5.168) and (5.169), respectively, and simultaneously belonging to domains of admissible controls of both systems, then

$$0 \le \Phi_\varepsilon(u_*) - \Phi_\varepsilon(u_{0\varepsilon}) \le C\varepsilon \quad \text{for} \quad \varepsilon \to 0. \tag{5.173}$$

Construct the control $u_{0\varepsilon}(\zeta, x)$. Equations of dynamic programming (5.159) in accepted notation are reduced to the form

$$V_\zeta^\varepsilon(\zeta,x) + \mathcal{L}_0 V^\varepsilon(\zeta,x) + H^u(\zeta/\varepsilon^2, x, V_x^\varepsilon(\zeta,x)) = \gamma, \tag{5.174}$$

where

$$H^u(t,x,q) = \min_{u\in U} H(t,x,u,q), \tag{5.175}$$

$$H(t,x,u,q)=(g(t,x,u),q)+\varphi(t,x,u). \qquad (5.176)$$

Let $\bar{V}^{\varepsilon}(\zeta,x)$ be a periodic (quasi-periodic) solution of Eq. (5.174). Then from (5.175) we get

$$u_{0\varepsilon}(\zeta,x)=U(\zeta/\varepsilon^2,x,\bar{V}_x^{\varepsilon}(\zeta,x)), \qquad (5.177)$$

In analogy with section 5.2, it can be shown that the control (5.177) is approximated by the expression

$$u_0(\zeta,x,\varepsilon)=U(\zeta/\varepsilon^2,x,V_x^0(x)), \qquad (5.178)$$

where $V^0(x)$ is a solution of the stationary equation

$$\mathcal{L}_0 V^0+H_0(x,V_x^0(\zeta,x))=\gamma^0, \qquad (5.179)$$

and

$$H^0(x,q)=\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0}H^u(t,x,q)dt. \qquad (5.180)$$

In this case, estimates

$$|\gamma^0-\Phi_\varepsilon(u_*)|\le C\varepsilon^2, \quad 0\le\Phi_\varepsilon(u_0)-\Phi_\varepsilon(u_*)\le C\varepsilon, \quad \varepsilon\to 0, \qquad (5.181)$$

hold true [145].

Inequalities (5.181) are proved as the analogous inequality (5.108). Theorem A.7 was used for the comparison of the functions $\bar{V}^{\varepsilon}(\zeta,x)$ and $V^0(x)$.

Let us give some examples for the proposed procedure.

Construct a control, minimizing a deviation form the equilibrium state in a linear system

$$\ddot{z}+[A+\varepsilon\Xi(t)]z+2\varepsilon^2\beta\dot{z}=\varepsilon F(t)+\varepsilon^2 Gu. \qquad (5.182)$$

Here the same notation is used as in the system ((5.110), β is a positively determined matrix of dissipation coefficients. Let us write the functional for the system in the form

$$\Phi_\varepsilon(u)=\lim_{T\to\infty}\frac{1}{T}\int_0^T M(R'\rho R+u'ru)dt. \qquad (5.183)$$

Here R is a vector of oscillation amplitudes with components

$$R_j=\left(z_j^2+\lambda_j^{-2}\dot{z}_j^2\right)^{1/2}, \quad j=1,\dots,n,$$

$\rho\ge 0$, $r>0$ are diagonal matrices of respective dimensions. In analogy with

Section 4.2, a quasi-optimal control $u_0 = \left(u_0^1, \ldots, u_0^m\right)$ is expressed by Eq. (5.120). Accounting for (5.97), (5.121), (5.179), we can write an averaged equation of dynamic programming in the form

$$\sum_{j=1}^{n}\left(v_j - \beta_{jj}\right)\frac{\partial V^0}{\partial y_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 V^0}{\partial y_i \partial y_j}$$

$$-\frac{1}{8}\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{g_{jk}^2 e^{-2y_j}}{\lambda_j^2 r_k}\left(\frac{\partial V^0}{\partial y_j}\right)^2 + \sum_{j=1}^{n}\rho_j e^{2y_j} = \gamma^0, \quad e^{y_j} = R_j. \tag{5.184}$$

Coefficients of Eq. (5.184) have the same sense as in (5.119), (5.121). The function V^0 should have the form

$$V^0(y) = \sum_{j=1}^{n} P_j e^{2y_j}, \tag{5.185}$$

where coefficients P_j satisfy a stationary condition of Riccati of the type (5.123)

$$2\left(v_l - \beta_{ll} + \alpha_{ll}\right)P_l + \sum_{\substack{j=1\\ j\neq l}}^{n} P_j q_{jl} - \frac{1}{2}P_l^2\sum_{j=1}^{n}\frac{g_{lj}^2}{r_j\lambda_l^2} + \rho_l = 0, \quad l = 1, \ldots, n, \tag{5.186}$$

and the functional value is determined by the relation

$$2\sum_{j=1}^{n} d_{jj}P_j = \gamma^0. \tag{5.187}$$

Here d_{jj} are the same coefficients as in ((4.50).

In its turn, an optimal control is presented in the form of a feedback (5.126) with constant coefficients. It follows from (5.186), (5.187), (5.126) that the feedback coefficients P_j depend only on characteristics of parametric disturbance, an external excitation affects only the value of the functional.

For a system with one degree of freedom, owing to (5.127), (5.128), we have

$$u^0 = -r\lambda^2 p\dot{z}, \tag{5.188}$$

where a coefficient p satisfies the equation

$$2(v + \alpha - \beta)p - \left(2r\lambda^2\right)^{-1} p^2 + \rho = 0, \quad v = \alpha = S^{\xi}(2\lambda)/8\lambda^2, \tag{5.189}$$

i.e.,

$$p = 2r\lambda^2\left[k + \left(k^2 + \rho/2r\lambda^2\right)^{1/2}\right], \quad k = 2\alpha = \beta. \tag{5.190}$$

In its turn,

$$\gamma^0 = 2d_{11}p = pS^{\zeta}(\lambda)/\lambda^2. \tag{5.191}$$

Till now we examined systems with disturbances without deterministic components. At the same time, all the results hold if a disturbance has periodic or quasi-periodic components.

Let us examine a problem of an optimal stabilization for a pendulum. An object is attached to an oscillating base by a linear-elastic link. An acceleration of the base is $w(t)$, α is an angle between the direction of the acceleration vector and the horizontal line, $w_x = w\cos\alpha$, $w_y = w\sin\alpha$. The system stabilization should be implemented by means of an additional moment $M(t)$ which diminishes a deviation of the pendant from the vertical. Suppose that the center of mass coincides with center of symmetry and that there are no rotation around the center of mass. In such a case the system can be treated as a mathematical

Motion equations of the pendulum have the form

$$m\ddot{z} - m(l+z)\dot{\varphi}^2 + 2c(z+\Delta) + 2b_1\dot{z} - mg\cos\varphi = = m\left[w_y(t)\cos\varphi + w_x(t)\sin\varphi\right], \tag{5.192}$$

$$m(l+z)\left[(l+z)\ddot{\varphi} + 2\dot{z}\dot{\varphi} + g\sin\varphi\right] + 2b_2\dot{\varphi} = = m(l+z)\left[w_x(t)\cos\varphi + w_y(t)\sin\varphi\right] + M(t).$$

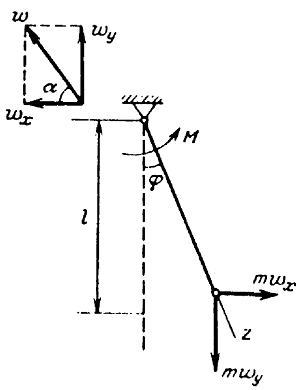


Fig. 5.1

Here m is a mass of the object, l is a distance between the center of mass to the base in a position of a static equilibrium, g is an acceleration due to gravity, c is a stiffness of the link, $\Delta = mg/(2c)$ is a static deformation of the link, b_1, b_2 are dissipation coefficients, φ is an angle of deviation from the vertical.

Suppose that the acceleration of the base $w(t)$ can be presented as a periodic process with the white noise. A narrow-band random processes with a marked carrier frequency can be described in this way [118]. Let, further, a frequency of the first excitation harmonic coincide with the frequency $\lambda_2 = (g/l)^{1/2}$ of angle oscillations of the pendulum. Considering, for the sake of brevity, that a periodic component of oscillations can be presented in a form of a cosine-expansion, we can write

$$w(t) = w^0(t) + h(t), \quad h(t) = h_0 + 2\sum_{k=1}^{\infty} h_k \cos k\lambda_2 t, \tag{5.193}$$

where $w^0(t)$ is a stationary random process with a zero mean and a spectral density $S_0 = \text{const}$.

It is also supposed that deviations of the pendulum from the vertical and link deformations are small, i.e., $|\varphi| << 1$, $|\delta| << 1$, $\delta = z/l$, $|\dot{\varphi}|/\lambda << 1$, $|\dot{\delta}|/\lambda << 1$, $\lambda = \min(\lambda_1, \lambda_2)$, $\lambda_1^2 = 2c/m$, $\lambda_2^2 = g/l$. So we can leave in our equations terms with an order not higher than two with respect to coordinates and velocities. Supposing a smallness of disturbing and controlling factors, we can re-write (5.192) in the form

$$\ddot{\delta} + \lambda_1^2\delta + 2\varepsilon^2\beta_1\dot{\delta} = \varepsilon(\xi_1(t) + \varepsilon\theta_1(t))\varphi + \varepsilon(\xi_2(t) + \varepsilon\theta_2(t)) + \varepsilon^2\left(\dot{\varphi}^2 - \frac{1}{2}\lambda_2^2\varphi^2\right),$$

$$\ddot{\varphi} + \lambda_2^2\varphi + 2\varepsilon^2\beta_1\dot{\varphi} = \varepsilon(1-\delta)(\xi_1(t) + \varepsilon\theta_1(t)) + \varepsilon(\xi_2(t) + \varepsilon\theta_2(t))$$
$$\varepsilon^2\left(2\dot{\delta}\dot{\varphi} - \lambda_2^2\varphi\delta\right) + \varepsilon^2 u. \tag{5.194}$$

Here ε is a small parameter,

$$\varepsilon\xi_1(t) = l^{-1}w^0(t)\cos\alpha, \quad \varepsilon\xi_2(t) = l^{-1}w^0(t)\sin\alpha,$$
$$\varepsilon^2\theta_1(t) = l^{-1}h(t)\cos\alpha, \quad \varepsilon^2\theta_2(t) = l^{-1}h(t)\sin\alpha, \tag{5.195}$$
$$\varepsilon^2\beta_1 = b_1/m, \quad \varepsilon^2\beta_2 = b_2/ml^2, \quad \varepsilon^2 u = 2M/ml^2.$$

Random $(\xi_j(t))$ and deterministic $(\theta_j(t))$ disturbances act in the same way on the system motion; in order to account for this fact, different exponents of the small parameter are introduced [compare (4.24)].

A control, minimizing a mean-square functional of the type (5.188), should be found; it can be assumed that $r = 1$.

A dependence of coefficients of Eq. (5.194) on time is determined not only by random disturbances, hence it is impossible to divide averaged equations for amplitudes and phases of oscillations. So, it is expedient to use the variable replacement of the type (4.59). By a replacement

$$\delta = x_1^1\cos\lambda_1 t + x_1^2\sin\lambda_1 t + \varepsilon^2\delta_0, \quad \varphi = x_2^1\cos\lambda_2 t + x_2^2\sin\lambda_2 t \tag{5.196}$$

we reduce (5.194) to the form [compare with (4.60), (4.61)]

$$\dot{x}^1 = \varepsilon\Lambda^{-1}F_s(t)\left[(\Xi(t) + \varepsilon\Theta(t))(F_c(t)x^1 + F_s(t)x^2) - \zeta(t) - \varepsilon\eta(t)\right]$$
$$+ 2\varepsilon^2F_s(t)\beta(-F_s(t)x^1 + F_c(t)x^2) - \varepsilon^2\Lambda^{-1}F_s(t)\left[Gu + Q(t, x^1, x^2)\right],$$

$$\dot{x}^2 = -\varepsilon\Lambda^{-1}F_c(t)\left[(\Xi(t) + \varepsilon\Theta(t))(F_c(t)x^1 + F_s(t)x^2) - \zeta(t) - \varepsilon\eta(t)\right] \tag{5.197}$$
$$- 2\varepsilon^2F_c(t)\beta(-F_s(t)x^1 + F_c(t)x^2) + \varepsilon^2\Lambda^{-1}F_c(t)\left[Gu + Q(t, x^1, x^2)\right],$$

$$\delta_0 = \frac{\lambda_2^2}{2\lambda_1^2}\left[\left(x_2^1\right)^2 + \left(x_2^2\right)^2\right].$$

Here matrices Λ, F_c, F_s are determined by Eqs. (4.59), $\beta = \operatorname{diag}\{\beta_1, \beta_2\}$, a vector $G = (0,1)$, Ξ is a matrix of random disturbances with components

$$\xi_{11} = 0, \quad \xi_{12} = \xi_{21} = -\xi_1, \quad \xi_{22} = \xi_2, \tag{5.198}$$

Θ is a matrix of periodic excitations with components

$$\theta_{11} = 0, \quad \theta_{12} = \theta_{21} = s_0 + \sum_{k=1}^{\infty} s_k \cos k\lambda_2 t,$$

$$\theta_{22} = r_0 + \sum_{k=1}^{\infty} r_k \cos k\lambda_2 t, \tag{5.199}$$

ζ is a vector of random disturbances,

$$\zeta_1 = \xi_2, \quad \zeta_2 = -\xi_1, \tag{5.200}$$

η is a vector of periodic excitations,

$$\eta_1 = \theta_{22}, \quad \eta_2 = \theta_{12}, \tag{5.201}$$

Q is a vector containing quadratic terms.

The problem functional in terms of the variables x^1, x^2 has the form

$$\Phi_\varepsilon(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T M\left[\left(x^1, \rho x^1\right) + \left(x^2, \rho x^2\right) + u^2\right]dt,$$

$$\rho = \operatorname{diag}\{\rho_1, \rho_2\}, \quad \rho_j \ge 0, \quad j = 1, 2. \tag{5.202}$$

Let us now construct a limit diffusion system. With the use of the same transformations as in (4.78), we get that in systems with a white-noise excitation coefficients $b^{ij} = 0$, and drift coefficients are formed only by means of averaging of deterministic components. As in Chapter 4, suppose that $\lambda_1/\lambda_2 \ne m/r$, m, $r = 1$, 2,... It is easy to show that quadratic links in this case in the first approximations do not cause additional „connected oscillations“, and the averaged equations are linear.

In analogy with (5.44), (4.88) we get

$$dx_{0\varepsilon}^1 = \left[-\beta x_{0\varepsilon}^1 + K^1 x_{0\varepsilon}^2 + d^1\left(\tau/\varepsilon^2\right)Gu\right]d\tau + s^{11}dw^1 + \sigma^{12}dw^2,$$

$$dx_{0\varepsilon}^2 = \left[K^2 x_{0\varepsilon}^1 - \beta x_{0\varepsilon}^2 + K + d^2\left(\tau/\varepsilon^2\right)Gu\right]d\tau + \sigma^{21}dw^1 + s^{22}dw^2. \tag{5.203}$$

Components of the matrices K^1, K^2 and of the vector K have the form $K_{11}^p =$ $= K_{12}^p = K_{21}^p = 0$, $p = 1, 2$.

$$K_{22}^1 = \lambda_2^{-1}(r_0 - r_2) = k^1, \quad K_{22}^2 = -\lambda_2^{-1}(r_0 + r_2) = k^2,$$
$$K_1 = 0, \quad K_2 = \lambda_2 s_1 = k. \tag{5.204}$$

Matrices σ^{pq}, s^{qq} are calculated according to Eqs. (4.64), (4.69), (4.89),

$$d^1(t) = -\Lambda^{-1}F_s(t), \quad d^2(t) = -\Lambda^{-1}F_c(t). \tag{5.205}$$

From (5.176), (5.183), (5.192) we have

$$H(t,x,u,q) = \lambda_2^{-1}\left(-q^1\sin\lambda_2 t + q^2\cos\lambda_2 t\right)u + u^2 + \left(x^1,\rho x^1\right) + \left(x^2,\rho x^2\right). \tag{5.206}$$

In the given case $q^r = \partial V/\partial x_2^r$, $r = 1, 2$. From the minimum condition

$$u = -(2\lambda_2)^{-1}\left(-q^1\sin\lambda_2 t + q^2\cos\lambda_2 t\right). \tag{5.207}$$

and

$$H^u = \left(x^1,\rho x^1\right) + \left(x^2,\rho x^2\right) - (4\lambda_2)^{-1}\left(-q^1\sin\lambda_2 t + q^2\cos\lambda_2 t\right). \tag{5.208}$$

Averaging (5.208), we get

$$H^0 = \left(x^1,\rho x^1\right) + \left(x^2,\rho x^2\right) - D_0\left[\left(q^1\right)^2 + \left(q^2\right)^2\right], \quad D_0 = 1/8\lambda_2^2. \tag{5.209}$$

An averaged equation of dynamic programming then get the form

$$\sum_{j=1}^{2}\left\{\sum_{k=1}^{2}\frac{1}{2}\left[A_{jk}^{11}\frac{\partial^2 V^0}{\partial x_j^1\partial x_k^1} + A_{jk}^{12}\frac{\partial^2 V^0}{\partial x_j^1\partial x_k^2} + A_{jk}^{21}\frac{\partial^2 V^0}{\partial x_j^2\partial x_k^1} + A_{jk}^{22}\frac{\partial^2 V^0}{\partial x_j^2\partial x_k^2}\right]\right.$$
$$\left. + \frac{1}{2}d_{jj}\left[\frac{\partial^2 V^0}{\left(\partial x_j^1\right)^2} + \frac{\partial^2 V^0}{\left(\partial x_j^2\right)^2}\right] - \beta_j\left(x_j^1\frac{\partial V^0}{\partial x_j^1} + x_j^2\frac{\partial V^0}{\partial x_j^2}\right) + \rho_j\left[\left(x_j^1\right)^2 + \left(x_j^2\right)^2\right]\right\}$$
$$- D_0\left[\left(\frac{\partial V^0}{\partial x_2^1}\right)^2 + \left(\frac{\partial V^0}{\partial x_2^2}\right)^2\right] + k^1 x_2^2\frac{\partial V^0}{\partial x_2^1} + k^2 x_2^1\frac{\partial V^0}{\partial x_2^2} + k\frac{\partial V^0}{\partial x_2^2} = \gamma^0. \tag{5.210}$$

Coefficients A_{jj}^{pq} are determined by Eqs. (4.64), (4.65), d_{jj} – by Eq. (4.50). Let us note that, owing to (5.193), in the problem under study $A_{11}^{pq} = 0$. A solution to Eq. (5.210) is constructed in the form

$$V^0\left(x^1,x^2\right) = \frac{1}{2}\sum_{j=1}^{2}\left[P_j^1\left(x_j^1\right)^2 + P_j^2\left(x_j^2\right)^2\right] + Qx_2^1x_2^2 + S^1x_2^1 + S^2x_2^2. \tag{5.211}$$

Substituting (5.211) into (5.210) and from the equality for coefficient in terms of the same order with respect to x_j^1, x_j^2, we will have

$$P_2^1 = P_1^2 = P_1, \quad q_{21}\left(P_2^1 + P_2^2\right) - \beta_1 P_1 + \rho_1 = 0, \tag{5.212}$$

$$2q_{12}P_1 + \left(\alpha_{22} + \eta_{22}\right)\left(P_2^1 + P_2^2\right) - D_0\left[\left(P_2^j\right)^2 + Q^2\right] - \beta_2 P_2^j + k^j Q + \rho_2 = 0,$$
$$j = 1, 2. \tag{5.213}$$

$$2Q\left(\alpha_{22} + \eta_{22}\right) - 2D_0 Q\left(P_2^1 + P_2^2\right) - 2\beta_2 Q + \left(k^1 + k^2\right)\left(P_2^1 + P_2^2\right) = 0, \tag{5.214}$$

$$\begin{gathered} -2D_0\left(P_2^1 S^1 + QS^2\right) + k^2 S^2 + kQ = 0, \\ -2D_0\left(P_2^2 S^2 + QS^1\right) + k^1 S^1 + kP_2^2 = 0. \end{gathered} \tag{5.215}$$

Coefficients q_{pq}, a_{pp}, η_{pp}, d_{pp} are calculated according to Eqs. (4.50), (4.65). Accounting for (5.198), we get

$$\begin{gathered} \alpha_{22} = \frac{1}{8\lambda_2^2} S_2, \quad \eta_{22} = \frac{3}{8\lambda_2^2} S_2, \quad q_{12} = q_{21} = \frac{1}{4\lambda_1\lambda_2} S_1, \\ d_{11} = \frac{1}{2\lambda_1^2} S_2, \quad d_{22} = \frac{1}{2\lambda_2^2} S_2, \end{gathered} \tag{5.216}$$

where, owing to (5.195), (5.198),

$$S_1 = l^{-2} S_0 \cos^2\alpha, \qquad S_2 = l^{-2} S_0 \sin^2\alpha. \tag{5.217}$$

The value of the functional is determined by the relation

$$\gamma^0 = \frac{1}{2}\sum_{j=1}^{2} d_{jj}\left(P_j^1 + P_j^2\right) - D_0\left[\left(S_j^1\right)^2 + \left(S_j^2\right)^2\right] + kS^2. \tag{5.218}$$

Substituting (5.211) into (5.207), we get

$$u_0 = \frac{1}{2\lambda_2}\left[-\left(P_2 x_2^1 + Qx_2^2 + S^1\right)\sin\lambda t + \left(P_2^2 x_2^2 + Qx_2^1 + S^2\right)\cos\lambda t\right]. \tag{5.219}$$

Taking into account for (5.186), we can write

$$u_0 = -\frac{1}{2\lambda_2}\left\{\frac{h}{\lambda_2}\dot\varphi - r\varphi + Q\left[\varphi\cos 2\lambda_2 t - \frac{1}{\lambda_2}\dot\varphi\sin 2\lambda_2 t\right] + S^1\sin\lambda_2 t + S^1\cos\lambda_2 t\right\} \tag{5.220}$$

$$h = \frac{1}{2}\left(P_2^1 + P_2^2\right), \qquad r = \frac{1}{2}\left(P_2^1 - P_2^2\right).$$

Let us analyze some partial cases.

1) $k^1 = k^2 = 0$ (the parametric disturbance includes only the first harmonic, there are no parametric resonance). In this case $Q = 0$, $S^1 = 0$,

$$P_2^1 = P_2^2 = h, \qquad r = 0, \qquad S^2 = k/2D_0 \tag{5.221}$$

and

$$u_0 = -\frac{1}{2\lambda_2^2} h\dot{\varphi} - 2s_1 \cos 2\lambda_2 t\,. \tag{5.222}$$

After substitution of (5.222) into (5.294), it is obvious that the second term in (5.222) entirely compensates the external disturbance. The coefficient h is determined from the solution of Eqs. (5.112), (5.113)

$$\begin{gathered} P_1 = \beta_1^{-1}(\rho_1 + 2q_{21})h, \\ h^2 - 2D_0 h(\alpha_{22} + \eta_{22}) - \beta_2/2 + q_{12}\beta_1^{-1}(\rho_1 + 2q_{21}) - (\rho_2 - 2\beta_1^{-1}\rho_1 q_{12}) = 0\,. \end{gathered} \tag{5.223}$$

The feedback is formed only with respect to $\dot{\varphi}$, but because of the linkage of the equations, the coefficient h and the value of the functional depend on parameters of the entire system.

2) $k = 0$ (the excitation has no first harmonic, there are no external resonance). Then $S^1 = S^2 = 0$.

3) There are no random disturbances. In this case we get the periodic optimization problem. In Eqs. (5.200) it should be considered that $A_{jk}^{pq} = 0$, $d_{jj} = 0$.

Eqs. (3.211) – (3.220) are kept, but the structure of Eqs. (3.212) – (3.214) changes. Let, for the sake of simplicity, consider that $\beta_2 = 0$ (assuming that dissipative forces are formed by the control). Then, from (5.213), (3.214), we get

$$Q = (2D_0)^{-1}(k_1 + k_2), \qquad P^j = -Q^2 + D_0^{-1}(k^\nu + \rho_2)\,. \tag{5.224}$$

Substituting (5.204), (5.209) into (5.224), we get

$$(P_2^{1,2})^2 = 8\lambda_2^2(\rho_2 \pm 8r_0 r_2), \qquad Q = -8\lambda_2^2 r_2\,. \tag{5.225}$$

Obviously, this problem has a solution when $\rho_2 > 8|r_0 r_2|$.

In the deterministic case, it is possible to write directly the equations of dynamic programming for the system (5.197) and to average them, omitting the stage of partial averaging [155]. The obtained results will then coincide with given ones. We can get the same results, using the averaging method in equations of the maximum principle (Section 3.2).

A Appendix

A.1 Pontryagin Maximum Principle

Let us give optimality conditions for oscillatory systems, dynamics of which is described by differential equations. A standard formulation of the optimal control problem is the following. Let $x(t) \in R_n$ be a vector–function of phase coordinates of the system. It is determined as a solution of a differential equation

$$\dot{x} = f(t, x, u), \quad t \in [0, T], \quad u \in R_m, \tag{A.1}$$

satisfying conditions on the ends of the interval

$$\varphi(t_0, x(t_0); T, x(T)) = 0, \tag{A.2}$$

where φ is an l–dimensional vector, $l \leq 2n+1$.

The most widespread case of the conditions (A.2) is an introduction of initial

$$x(t_0) = x_0 \tag{A.3}$$

and boundary conditions

$$\psi(t_f, x(t_f)) = 0. \tag{A.4}$$

An optimization problem with control constrains can be formulated in the following way.

It is necessary to find a piecewise continuous control $u \in R_m$ satisfying the constraints

$$u(t) \in U, \quad U \subset R_m, \tag{A.5}$$

and minimizing a functional

$$\Phi = g[t_0, x(t_0); t_f, x(t_f)] + \int_{t_0}^{t_f} f_0(t, x, u)dt \tag{A.6}$$

on trajectories of the system (A.1), (A.2).

Let us apply the maximum principle of L.S. Pontryagin [114], which determines the necessary conditions of optimality, to the formulated problem.

Introduce additional scalar functions

$$H(t,x,u,p) = -p_0 f_0(t,x,u) + p'(t) f(t,x,u) \tag{A.7}$$

(a Hamilton function) and

$$h(t_0,x(t_0);t_f,x(t_f)) = g(t_0,x(t_0);t_f,x(t_f)) + \rho'\varphi(t_0,x(t_0);t_f,x(t_f)). \tag{A.8}$$

Here $p(t)$ is an n–dimensional vector of Lagrange multipliers, ρ is a constant l–dimensional vector, p_0 is a constant multiplier. It is supposed that functions f, f_0 are piecewise continuous with respect to t and have continuous partial derivatives up to the second order inclusive with respect to other arguments in domains of the space of problem variables; functions g and φ are continuous and two times continuously differentiable with respect to all their variables.

A control $\tilde{u}(t) \in U$ is called *admissible* if for $u = \tilde{u}(t)$ there exists a unique solution of Eq. (A.1) with boundary conditions (A.2).

The optimal control belongs to a class of admissible controls.

The maximum principle of L.S. Pontryagin [114, 125]. *If a control* $u_*(t)$ *and a trajectory* $x_*(t)$ *provide a minimum of the functional* (A.6) *for* Eq. (A.1), *control constrains* (A.5) *and boundary conditions* (A.2), *then there exist such continuous vector–function* $p(t)$, *a constant* $p_0 \geq 0$ *and a constant vector* ρ *that for each* $t \in [t_0, T]$ *the Hamilton function* $H_*(t,x,u,p)$ *gets in a point* u_* *the maximum value with respect to all* $u \in U$

$$H(t,x_*,u_*,p) = \max_{u \in U} H(t,x,u,p), \tag{A.9}$$

where the vector $p(t)$ *is determined by the equation*

$$\dot{p} = -\partial H/\partial x \tag{A.10}$$

and by boundary conditions

$$p(t_0) = \partial h/\partial x(t_0), \quad p(t_f) = -\partial h/\partial x(t_{f0}), \tag{A.11}$$

$$\partial h/\partial t_0 + H(t_0) = 0, \quad \partial h/\partial t_f - H(t_f) = 0, \tag{A.12}$$

where $H(s) = H(s,x_*(s),u_*(s),p(s))$.

If $u \in \operatorname{int} U$ or the domain U coincides with the entire space then the condition (A.9) is reduced to the form

$$\partial H/\partial u = 0. \tag{A.13}$$

Conditions (A.9), (A.13) allow us to express u as a function of phase and adjoint variables

$$u_* = U(t,x,p). \tag{A.14}$$

Let us formulate the boundary conditions of the maximum principle for some particular cases.

A case of the boundary conditions (A.3) with a free right-hand end of a trajectory is widespread. Here the maximum principle reduces a solution of an optimization problem to the solution of the boundary–value problem

$$\dot{x} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial x, \tag{A.15}$$

$$x(t_0) = x_0, \qquad p(t_f) = 0. \tag{A.16}$$

If the function (A.8) depends only on t_f, $x(t_f)$ then the boundary conditions for p have the form

$$p(t_f) = -h_x(t_f, x(t_f)). \tag{A.17}$$

In a problem of a high-speed action it is usually necessary to minimize the functional

$$\Phi = g[t_f, x(t_f)] + \int_{t_0}^{t_f} f_0(t, x, u)dt, \tag{A.18}$$

and the moment $t_f = t_*$ for the fixed condition (A.4) for the right-hand end of the trajectory. The boundary–value problem (A.15), (A.7) and an additional condition

$$\partial h/\partial t_f = H(t_f), \qquad t_f = t_*, \tag{A.19}$$

serve for a determination of the optimal control (A.14) and of the moment t_*. In particular, if $\psi = \psi(x(t_f))$, $g = 0$, then the condition (A.12) has the form

$$H(t_f) = 0. \tag{A.20}$$

Let us consider another equality

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \left(\frac{\partial H}{\partial x}, \dot{x}\right) + \left(\frac{\partial H}{\partial p}, \dot{p}\right) = \frac{\partial H}{\partial t} + \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p}\right) - \left(\frac{\partial H}{\partial p}, \frac{\partial H}{\partial x}\right) = \frac{\partial H}{\partial t}. \tag{A.21}$$

It shows that in an autonomous system the relation $dH/dt = 0$ is fulfilled for $\partial H/\partial t = 0$, and Eqs. (A.15) have the first integral

$$H = \text{const}. \tag{A.22}$$

In Section 2.3, the control problems for systems with impacts were examined. The velocity of the striking (working) element undergoes a discontinuity, and discontinuity conditions for velocity at the impact moment supplement the periodicity conditions. In a general case, this problem can be formulated as follows [31, 125].

Let a trajectory of the system (A.1), (A.2) have discontinuities of the first kind at moments $t = t_f$, $i = 1, \ldots, q$:

$$\eta_j\left[x(t_i - 0), x(t_i + 0), t_i\right] = 0, \qquad j = 1, \ldots, q. \tag{A.23}$$

Then the optimal control $u_*(t)$, providing a minimum of the functional (A.6), is determined by relations (A.9) – (A.12), by supplemented conditions (A.23) and by discontinuity conditions

$$p(t_i - 0) = \partial\eta/\partial x(t_i - 0), \qquad p(t_i + 0) = -\partial\eta/\partial x(t_i + 0),$$

$$(H)_{t_i+0} - (H)_{t_i-0} = -\partial\eta/\partial t_i, \qquad \eta = \sum_{j=1}^{a} v_j \eta_j, \tag{A.24}$$

where v is a constant vector.

An analogous reduction to a boundary–value problem can be also done for other problems of the optimal control.

In the optimal control theory, a considerable attention is paid to the existence conditions of the control providing a minimum of the functional, sufficient optimality conditions, etc.

It is supposed, as a rule, that an optimal control exists and is determined by the given solution of the boundary–value problem of the maximum principle.

A.2 Disturbances in Optimal Systems

In the main part of this book, the systems with dynamics described by differential equations as well as by integral ones are examined. Write for the generality an equation of a disturbed system in the form

$$x = A(t, x, u, \varepsilon), \tag{A.25}$$

where A is some operator, u is a control, ε is a small parameter. Let V_ε be a set of admissible controls of the system (A.25), such that for any $u \in V_\varepsilon$ a solution $X_\varepsilon^u(t)$ of the system (A.25) exists and is unique.

Let further exist a generating system

$$x = A(t, x, u, 0). \tag{A.26}$$

Let V_0 be a set of admissible controls of the system (A.26) such that for any $u \in V_0$ a solution $X_0^u(t)$ of the system (A.26) exists and is unique.

Denote by $\Phi\left(X_\varepsilon^u(t)\right) = \Phi_\varepsilon(u)$ some continuous functional which is determined on trajectories of the disturbed system (A.25) for a fixed control $u \in V_\varepsilon$. Let

$\Phi\left(X_0^u(t)\right) = \Phi_0(u)$ be the same functional, determined on trajectories of the system (A.26) for $u \in V_0$.

Denote u_* as an optimal control providing the minimum of the functional $\Phi_\varepsilon(u)$ with a constraint $u \in U \subset V_\varepsilon \cap V_0$:

$$u_* = \arg\min_{u \in U} \Phi_\varepsilon(u). \tag{A.27}$$

Analogously,

$$u_0 = \arg\min_{u \in U} \Phi_0(u), \tag{A.28}$$

i.e., u_0 is a control minimizing the same functional on trajectories of the system (A.26) with the same constraint $u \in U \subset V_\varepsilon \cap V_0$.

Suppose that for $\varepsilon \to 0$ and any $u \in V_\varepsilon \cap V_0$ the solution $X_\varepsilon^u(t)$ of the disturbed system (A.25) weakly converges to the solution $X_0^u(t)$ of the generating system (A.26), i.e., for any continuous functional

$$\left|\Phi_\varepsilon(u) - \Phi_0(u)\right| \le C_\varepsilon, \quad C_\varepsilon \to 0 \quad \text{for} \quad \varepsilon \to 0. \tag{A.29}$$

Then the control (A.28) is quasi-optimal with respect to the disturbed system with an estimate of the same order of magnitude, namely,

$$0 \le \Phi_\varepsilon(u_0) - \Phi_\varepsilon(u_*) \le 2C_\varepsilon. \tag{A.30}$$

The left-hand part of the inequality (A.20) follows directly from the optimality condition (A.27). Let us prove the upper estimate. Re-write (A.30) in the form

$$\Phi_\varepsilon(u_0) - \Phi_\varepsilon(u_*) = \left[\Phi_\varepsilon(u_0) - \Phi_0(u_0)\right] + \left[\Phi_0(u_0) - \Phi_\varepsilon(u_*)\right]. \tag{A.31}$$

The first term in (A.31) corresponds to the condition (A.29). In order to estimate the second term, let us write the obvious inequality

$$\Phi_0(u_0) \le \Phi_\varepsilon(u_*) + \left[\Phi_0(u_*) - \Phi_\varepsilon(u_*)\right],$$

and, owing to (A.29),

$$\Phi_0(u_*) - \Phi_\varepsilon(u_*) \le C_\varepsilon. \tag{A.32}$$

Substituting (A.32) into (A.31), we get the inequality (A.30).

The problems of an optimal control existence in systems close to generating systems are discussed, for instance, in [156] for deterministic systems and in [180, 181] for stochastic ones.

A.3
Main Theorems of the Averaging Method

In this Section asymptotic methods based on averaging of the right-hand parts of differential equations with respect to some of the variables are given. These methods, called *averaging methods* or *methods of movement separation*, serve as a main tool for analysis of equations of the form

$$\dot{x} = \varepsilon X(x, \psi, \varepsilon),$$
$$\dot{\psi} = Y_0(x, \psi) + \varepsilon Y(x, \psi, \varepsilon),$$

where x and ψ are some vectors, ε is a small parameter [33, 38, 102, 103, 105]. Without concretization of the physics of such phenomena, let us note that such equations are typical for a wide class of problems of the oscillation theory.

Below we will limit our considerations only to main conclusions for equations of the kind

$$\dot{x} = \varepsilon X(t, \varepsilon), \tag{A.33}$$

where x and X are n–dimensional vectors. Systems of the form (A.33) are called „*systems in the standard form*".

Let in the domain

$$D{:}\{t \in [0, \infty), x \in S \subset R_n\} \tag{A.34}$$

the function $X(t, x)$ be continuously bounded, continuous with respect to t and satisfy the Lipschitz condition

$$|X(t, x_2) - X(t, x_1)| \le L|x_2 - x_1|, \tag{A.35}$$

with a constant independent of t. Consider further that for each $x \in S$ there exists the mean value with respect to time

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x) dt. \tag{A.36}$$

Then parallel with Eq. (A.33), an equation

$$dx_0/d\tau = X_0(x_0), \quad \tau = \varepsilon t, \tag{A.37}$$

is considered. It follows form (A.36) that the function $X_0(x)$ satisfies the condition (A.35).

Theorem 3.1 (the first theorem of N.N. Bogolyubov) states a linkage between $x_0(\tau) = x_0(\varepsilon t)$ and a solution $x(t, \varepsilon)$ of Eq. (A.33), satisfying the condition

$$x_0(0) = x(0, \varepsilon) = a. \tag{A.38}$$

Theorem A.1 [102]. *Let in the domain D the function* $X(t,x)$ *be continuously bounded, continuous with respect to t and satisfy the Lipschitz condition* (A.35). *Let further the mean value with respect to time* (A.36) *exist for each* $x \in S$.

If the solution $x_0(\tau)$ *of* Eq. (A.35) *for* $0 \le \tau \le T_0$ *is in S with its environment then for any* $\eta > 0$ *there exists such* $\varepsilon_0 > 0$ *that for* $0 < \varepsilon < \varepsilon_0$ *the solution* $x(t,\varepsilon)$ *of Eq. (A.33) satisfying the condition* (A.38) *has an estimate*

$$|x(t,\varepsilon) - x(\varepsilon t)| \le \eta, \quad 0 \le \tau \le T_0/\varepsilon. \tag{A.39}$$

Let us consider the more general than (A.33) equation

$$\frac{dx}{d\tau} = X(\tau,x;\varepsilon), \quad 0 \le \tau \le T_0, \quad 0 \le \varepsilon \le \varepsilon_1, \tag{A.40}$$

with the function $X(\tau,x;\varepsilon)$ satisfying the Lipschitz condition with respect to x and the condition

$$\lim_{\varepsilon \to 0} \int_0^{\tau_0} X(\tau,x;\varepsilon)d\tau = \int_0^{\tau_0} X(\tau,x;0)d\tau, \quad 0 \le \tau_0 \le T_0, \tag{A.41}$$

which is called the *condition of integral continuity* for $\varepsilon = 0$.

Eq. (A.40) is reduced to (A.33) if it is considered that

$$X(\tau,x;\varepsilon) = \begin{cases} X(\tau/\varepsilon, x), \varepsilon > 0, \\ X_0(x), \varepsilon = 0, \end{cases} \tag{A.42}$$

and with the replacement $\tau = \varepsilon t$.

The condition (A.41) is fulfilled in this case since

$$\lim_{\varepsilon \to 0} \int_0^{\tau_0} X(\tau/\varepsilon, x)d\tau = \tau_0 X_0(x),$$

or, in other way,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\tau_0} \int_0^{\tau_0} X(t,x)d\tau = X_0(x), \tag{A.43}$$

and the latter is equivalent to (A.36).

Theorem A.1 follows form the more general theorem of M.A. Krasnoselsky and M.G. Krein about the continuous dependence of solutions of differential equations on their parameters.

Theorem A.2 [91]. *Let*

1) a function $X(\tau,x,\varepsilon)$ *be determined* for $\tau \in [0,T_0]$, $x \in S$, $0 \le \varepsilon < \varepsilon_1$, *continuously bounded for these values of variables, piecewise continuous with respect* *to τ, integral–continuous for $\varepsilon = 0$ and satisfy Lipschitz condition with respect to x with a constant independent of τ, x;*

2) Eq. (3.40) *for $\varepsilon = 0$ have a solution $x_0(\tau) = x(\tau, 0)$ satisfying the condition*

$$x_0(\tau) \in \operatorname{int} S, \quad 0 \le \tau \le T_0.$$

Then for each $\eta > 0$ there exists ε_0, $0 < \varepsilon_0 < \varepsilon_1$, such that any solution $x(\tau, \varepsilon)$ of Eq. (A.40), *determined over the interval $[0, T_0]$ and satisfying the condition*

$$x_0(0; \varepsilon) = x_0(0) = a,$$

for $0 < \varepsilon < \varepsilon_0$ has an estimate

$$|x(\tau, \varepsilon) - x_0(\tau)| \le \eta, \quad 0 \le \tau \le T_0. \tag{A.44}$$

One of the most important conclusions of the averaging method states a correspondence between properties of solutions of exact and averaged equations for an infinite time interval.

Let again

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x) dt.$$

Suppose that for some point $\xi_0 \in \operatorname{int} S$

$$X_0(\xi_0) = 0, \tag{A.45}$$

i.e., the averaged equation (A.5) has a quasi-static solution

$$x_0(\tau) = \xi_0 = \text{const}.$$

Let us give the conditions for which the initial equation (A.33) for sufficiently small $\varepsilon > 0$ has a solution, bounded over the entire axis and which does not leave a small zone around the point ξ_0.

Theorem A.3 [102] (the second theorem of N.N. Bogolyubov) *Let a function $X(t, x)$ of the equation*

$$\frac{dx}{dt} = \varepsilon X(t, x)$$

satisfy the following conditions:

1) *averaged equations*

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi)$$

have a quasi-static solution $\xi = \xi_0$;

2) *real parts of all n roots of the characteristic equation*

$$\det\left|pI - X'_{0\xi}(\xi_0)\right| = 0 \tag{A.46}$$

for an equation with respect to variations

$$\frac{d\delta}{dt} = \varepsilon X'_{0\xi}(\xi_0)\delta$$

which corresponds to the quasi-static solution, differ from zero;

3) *the function* $X(t,x)$ *and its partial derivatives* $X_x(t,x)$ *are bounded and uniformly continuous in the domain*

$$D_1:\ \left[t,x \,/\, t \in (-\infty,\infty), x \in S_\rho(\xi_0)\right],$$

where $S_\rho(\xi_0)$ *is a* $\rho-$ *zone of the point* $\xi_0 \in \operatorname{int} S$ *;*

4) *the limit*

$$X_0(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} X(t,x)dt$$

exists uniformly with respect to $t \in (-\infty,\infty)$, $x \in S_\rho$.

Then for sufficiently small $\rho_0 > 0$ *there exists such a number* $\varepsilon_0 > 0$ *that for all* $0 < \varepsilon \le \varepsilon_0$ Eq. (A.33) *has a unique, bounded over the entire axis solution* $x^*(t,\varepsilon)$ *satisfying the condition*

$$\sup_{-\infty<t<\infty}\left|x^*(t,\varepsilon) - \xi_0\right| < \rho_0 . \tag{A.47}$$

If functions $X(t,x)$ and $X'_x(t,x)$ are nearly periodic (periodic) with respect to t then the solution $x^*(t,\varepsilon)$ is nearly periodic (periodic) with the same basis [34].

A subsequent development of the averaging method is connected with a broadening of the class of problems for which this method can be applied, and with a weakening of continuity conditions imposed on the right–hand parts of Eqs. (3.1).

In works [147, 148] a necessity of an application of the averaging method to systems of differential equations with discontinuities of the first kind with respect to fast variables was stated. In [33] a generalization of Theorems A.1, A.3 for systems with moment impulses causing a trajectory discontinuity is given. Thus, the correctness of the averaging scheme for systems studied in Section 3.3 is proved.

One partial scheme of the averaging method – a partial averaging – is convenient for optimal control problems [111 – 113]. The averaging can be carried out only for terms independent of the control, and then it is necessary to find the control minimizing the functional on trajectories of the partially averaged system. In such a case, a main structure of the system is singled out and the control construction simplifies.

The partial averaging technique is given in detail in [111, 112]. Here we give only the main results.

Let the system motion be described by an equation

$$\dot{x} = \varepsilon\left[X(t,x) + Y(t,x)\right], \tag{A.48}$$

$x \in R_n$, and let the mean value

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)dt \tag{A.49}$$

exist. Let us correspond a partially averaged system

$$\dot{x}_0 = \varepsilon\left[X_0(x_0) + Y(t,x_0)\right], \tag{A.50}$$

with the same initial conditions

$$x(0,\varepsilon) = x_0(0,\varepsilon) = a\,. \tag{A.51}$$

to the system (A.48).

Theorem A.4 [111]. *Let in the domain*

$$D\!:\left\{t \ge 0, x \in S \subset R_n\right\}$$

the following conditions be satisfied:

1) *the functions* $X(t,x)$ *and* $Y(t,x)$ *are uniformly bounded and piecewise continuous with respect to t and satisfy the Lipschitz condition with respect to x with a constant independent of t;*

2) *the mean value* (A.49) *exists uniformly with respect to* $x \in S$;

3) *a solution* $x_0(t,\varepsilon)$ *of the averaged system* (A.50) *with the initial condition* (A.51) *is in the domain S together with its* ρ *– zone for all* $0 \le t \le T_0\varepsilon^{-1}$.

Then for each $\eta > 0$ *such* $\varepsilon_0 > 0$ *can be found that for* $0 < \varepsilon \le \varepsilon_0$ *an estimate*

$$\left|x(t,\varepsilon) - x_0(t,\varepsilon)\right| < \eta \tag{A.52}$$

holds true.

Theorem 3.4 allows also the further generalization linked with the weakening of continuity conditions and application of results to the infinite time interval.

The study of limit possibilities of control systems based on the maximum principle needs a solution of the boundary–value problem. It is shown in [6, 111, 134] that in periodic control problems the system of equations of the maximum principle can be written in the form

$$\frac{dx}{dt} = \varepsilon X(t,x), \qquad x = x(t,\varepsilon) \in R_n, \tag{A.53}$$

$$R\left[x(0,\varepsilon), x(t_f,\varepsilon)\right] = 0, \qquad t_f = O(\varepsilon^{-1}). \tag{A.54}$$

Eq. (A.54) unifies the boundary conditions for the both ends of the interval.

Let us compare the solution $x(t,x)$ of the system (A.53), (A.54) with the solution of the averaged system

$$\frac{dx_0}{d\tau} = X_0(x), \qquad \tau = \varepsilon t, \tag{A.55}$$

$$R\left[x(0), x(t_f)\right] = 0, \qquad \tau_f = \varepsilon t_f. \tag{A.56}$$

Let us underline that, on the whole, the initial conditions are not fixed with respect to the vectors x and x_0 and, generally speaking, $x(0,\varepsilon) \neq x_0(0)$.

A substantiation of the averaging scheme for the boundary–value problems is given in [6, 110, 111, 134] for various assumptions for the smoothness of the functions $X(t,x)$. Let use conditions of the book [6] as the most general.

Theorem A.5. *Let*

1) *the function* $X(t,x)$ *be defined for* $t \geq 0$ *and measurable with respect to t uniformly with respect to* $x \in S$ *, where S is an open domain in* R_n;

2) *the function* $X(t,x)$ *be periodic or uniformly quasi-periodic with respect to t;*

3) *functions* $X(t,x)$, $R(y,z)$ *be defined for all x, y,* $z \in S$ *and uniformly continuous with respect to x, y, z;*

4) *there exist such constants* $c_{1,2}$, $\lambda_{1,2}$ *that for* $t \geq 0$, *x, y,* $z \in S$

$$|X(t,x)| \leq c_1, \qquad |R(y,z)| \leq c_2,$$

$$|X_x(t,x)| \leq \lambda_1, \qquad |R_y(y,z)| \leq \lambda_2, \qquad |R_z(y,z)| \leq \lambda_2;$$

5) *the boundary-value problem* (A.53), (A.54) *for* $0 \leq \tau \leq \tau_f$ *have a unique solution* $x_0(\tau) \in S$.

Then for sufficiently small $\varepsilon \in (0,\varepsilon_0]$ *the solution* $x_0(\tau)$ *of the initial boundary-value problem* (A.53), (A.54) *is in the* $\varepsilon-$ *zone of the generating solution* $x(t,\varepsilon)$ *of the averaged problem* (A.55), (A.56)

$$|x(t,\varepsilon) - x_0(\tau)| \leq c\varepsilon, \qquad 0 \leq \tau \leq \tau_f. \tag{A.57}$$

$$|x(0,\varepsilon) - x_0(0)| \leq c\varepsilon, \qquad c = \text{const}. \tag{A.58}$$

It is shown in [6] that conditions 1) – 4) are necessary only in order to provide the closeness of the Cauchy problem for an averaged problem (A.53), (A.55) for arbitrary initial conditions. Thus, the constraints imposed on the right-hand part of the system (A.53) can be replaced by a more general condition of existence and closeness of solution of the Cauchy problem for initial and averaged systems.

Theorem 3.5a. *Let in the domain*

$$D:\{\tau \in [0,T], x \in S \subset R_n, 0 < \varepsilon \le \varepsilon_0\}$$

1) *for all initial conditions* $a \in S_a \subset S$ *a solution* $x(\tau,a,\varepsilon)$ *of the system* (A.21) *depend continuously on initial conditions, i.e.,*

$$x(0,a,\varepsilon) = a,$$

$$\lim_{a \to a_0} x(\tau,a,\varepsilon) = x(\tau,a_0,\varepsilon)$$

continuously with respect to τ, ε;

2) *for any initial conditions* $x_0(0) = a \in S_a$ *a solution* $x_0(\tau,a_0)$ *of the averaged system* (A.55) *be determined for all* τ *and its environment be in S;*

3) *the solution* $x(\tau,a,\varepsilon)$ *of the initial system (A.53) be approximated uniformly with respect to* τ, ε *by the solution* $x_0(\tau,a_0)$ *of the system* (A.55) *with the same initial conditions, i.e.,*

$$|x(\tau,a,\varepsilon) - x_0(\tau,a)| \le C\varepsilon, \quad x(0,a,\varepsilon) = x_0(0,a) = a;$$

4) *the function* $R(y,z)$ *be bonded and uniformly continuous with respect to y, z for all y,* $z \in S$ *with its derivatives*

$$|\partial R/\partial y| \le C, \quad |\partial R/\partial z| \le C;$$

5) *an equation*

$$R[x_0(0), x_0(T, x_0(0))] = 0$$

have a solution

$$x_0(0) = a_0 \in \operatorname{int} S_a$$

with respect to $x_0(0)$, *i.e.,*

$$R(a_0) = R[a_0, x_0(T,a)] = 0,$$

and the root a_0 *be simple, and* $\det|\partial R(a_0)/\partial a_0| \ne 0$.

Then for sufficiently small ε *the estimates* (A.57), (A.58) *hold true.*

Let us note that conditions 1), 3) of Theorem A.5a are satisfied both for smooth and discontinuous solutions (in particular, for solutions of motion equations for vibroimpact systems).

At last, any system close to (A.53) in the sense of the solution closeness (condition 3) can be examined instead of the system (A.55).

Theorems A.1 – A.5 embrace only the general information about the averaging method which is widely used below for a solution of optimization problems for deterministic systems.

Optimal synthesis problems in stochastic systems are reduced to the solution of non-linear equations in partial derivatives. In conclusion, let us give the main results of an application of the averaging principle to differential equations of the parabolic type [55,56, 92,93, 98, 119, 145, 146].

Consider the Cauchy problem for an equation

$$V_\tau^\varepsilon = \mathcal{L}_0 V^\varepsilon + f\left(\tau/\varepsilon, x, V^\varepsilon, V_x^\varepsilon\right) \tag{A.59}$$

with a boundary condition

$$V^\varepsilon(0,x) = \varphi(x). \tag{A.60}$$

Here

$$\mathcal{L}_0 = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j}. \tag{A.61}$$

The main part of results on averaging in equations of the type (A.59) concerns either linear equations [130, 146] or equations in which the function f possesses definite properties of differentiation [98, 119]; this limitations are discussed in detail in [55, 56].

In equations of dynamic programming, the coefficients may not satisfy differentiability conditions or possess a necessary smoothness. In order to estimate the limit transition, let us apply to Eqs. (A.59), (A.60) the results obtained in [92, 93] for equations of a more general form.

Suppose that there exists a solution $V^\varepsilon(\tau,x)$, bounded in any compact set, with a bounded derivative $V_x^\varepsilon(\tau,x)$ (the boundedness is understand for some norms; existence conditions of such a solution are stated, for instance, in [95] and do not depend on continuity properties of coefficients). Suppose that

1) a function $\varphi(x)$ is continuous and two times differentiable for $x \in R_n$;

2) functions $a(x)$, $b(x)$ are continuous and continuously differentiable for $x \in R_n$;

3) the condition of a uniform ellipticity is fulfilled; for all $x \in K$, $\xi \in R_n$

$$\lambda_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda_1|\xi|^2,$$

where λ_0, $\lambda_1 = \text{const} > 0$, K is a compact set in R_n.

Let also in the domain $D = I_t \times Q$, where Q is a compact set in $R_n \times R_1 \times R_n$,

$I_t = (-\infty, \infty)$:

4) a function $f(t,x,v,p)$ be uniformly bounded and continuous with respect to x, v, p uniformly with respect to other variables;

5) a limit

$$f^0(x,v,p) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T+t_0} f(t,x,v,p)dt \tag{A.62}$$

exist uniformly with respect to $t_0 \geq 0$, x, v, $p \in Q$;

6) a unique solution of the Cauchy problem for an averaged equation

$$V_\tau^0 = \mathcal{L}_0 V^0 + f(x,V^0,V_x^0), \quad V^0(0,x) = \varphi(x), \tag{A.63}$$

exist.

Theorem A.6. *If assumptions* 1) – 6) *are fulfilled then*

$$\lim_{\varepsilon\to 0} \left| V^\varepsilon(\tau,x) - V^0(\tau,x) \right| = 0,$$
$$\lim_{\varepsilon\to 0} \left| V_x^\varepsilon(\tau,x) - V_x^0(\tau,x) \right| = 0 \tag{A.64}$$

uniformly with respect to τ, x *for* $0 \leq \tau \leq \tau_f < \infty$, $x \in K$, K *is a compact set in* R_n.

Theorem A.7. *Let*

1) *the functions* $a(x)$, $b(x)$ *satisfy conditions* 1) - 3), 5) *of* Theorem A.6;

2) *the function* $f(t,x,v,p)$ *be measurable and periodic (or quasi-periodic) with respect to t, continuous with respect x, v, p and continuously differentiable with respect to v, p uniformly with respect to other variables from D;*

3) *an averaged elliptic equation*

$$\mathcal{L}_0 V^0(x) + f(x,V^0,V_x^0) = 0, \tag{A.65}$$

have a unique solution $\overline{V}^0$ *for which an operator of the first linear approximation has no spectrum for an imaginary axis.*

Then there exists a unique, bounded over the entire axis solution $\overline{V}^\varepsilon(\tau,x)$ *of* Eq. (A.59) *for which the estimates*

$$\lim_{\varepsilon\to 0} \left| \overline{V}^\varepsilon(\tau,x) - \overline{V}^0(\tau,x) \right| = 0,$$
$$\lim_{\varepsilon\to 0} \left| \overline{V}_x^\varepsilon(\tau,x) - \overline{V}_x^0(\tau,x) \right| = 0 \tag{A.66}$$

hold true.

An effective method for construction of successive approximations and estimates for convergence rate based on the method of multi-scale expansions [105] is suggested in [26].

A.4 Necessary Condition of the Optimality of Periodic Regimes

Let us give without a proof the main results determining minimum conditions for the functional

$$\Phi(u)=\frac{1}{T}\int_0^T f_0(t,x,u)dt\,, \qquad u\in U\,, \tag{A.67}$$

on T-periodic trajectories of the system

$$\dot{x}=f(t,x,u)\,, \qquad x(0)=x(T)\,. \tag{A.68}$$

Here U is a compact set in R_m, admissible controls $u(t)$ are T-periodic functions with respect to t with values from $U\big(u(t)\in U\big)$ providing a uniqueness of the periodic solution of Eq. (A.68). It is supposed that for $u\in U$, $x\in R_n$ functions f, f_0 are piecewise continuous and T-periodic with respect to t and continuous and continuously differentiable with respect to x and uniformly with respect to $t\in(-\infty,\infty)$.

Necessary optimality conditions for periodic regimes are formulated in [120, 160, 163, 164] and some other works; a review of results is given in [163, 185, 187]. In [187] the existence conditions for an optimal control are also given.

Let us now formulate optimality conditions following the results of [123].

Theorem A. 8. *Let $u_*(t)$ be an optimal periodic control, $x_*(t)$ be a respective optimal T-periodic trajectory and let an equation in variations*

$$\dot{\xi}=F_*(t)\xi\,, \qquad F_*=f_x\big(t,x_*(t),u_*(t)\big)$$

have no T-periodic solutions other than a trivial one for $u=u_*$, $x=x_*$ (condition A).

Then the maximum condition

$$H\big(t,x_*,u_*,p\big)=\max_{u\in U} H\big(t,x,u,p\big)\,, \tag{A.69}$$

where

$$H\big(t,x,u,p\big)=p'f(t,x,u)-f_0(t,x,u)\,, \tag{A.70}$$

and $p(t)$ is a T-periodic solution of the equation

$$\dot{p}=-H_x\big(t,x_*,u_*,p\big)\,, \qquad p(0)=p(T)\,, \tag{A.71}$$

holds true.

Condition A provides the existence and uniqueness of the T-periodic solution of Eq. (A.71).

If the system is autonomous

$$\dot{x} = f(x,u), \quad \Phi(u) = \frac{1}{T}\int_0^T f_0(x,u)dt, \tag{A.72}$$

then the period $T \in [T_{\min}, T_{\max}]$ should be determined from optimality conditions.

Theorem A.9. *Let there exist an optimal control* (x_*, u_*, T_*) *of the problem* (A.72) *and* condition A *be satisfied for a* T_**-periodic solution* (x_*, u_*).

Then the maximum condition

$$H(x_*, u_*, p, T_*) = \max_{u \in U} H(t, x, u, p) \tag{A.73}$$

is fulfilled, where

$$H(x,u,p,T) = p'f(x,u) - T^{-1} f_0(x,u), \tag{A.74}$$

and $p(t)$ *is a* T_**-periodic solution of the equation*

$$\dot{p} = -H_x(x_*, u_*, p, T_*); \quad p(0) = p(T_*). \tag{A.75}$$

If $T_* \in (T_{\min}, T_{\max})$ *then*

$$\Phi_* = \Phi(u_*) = \int_0^{T_*} p'(t) f(x_*(t), u_*(t)) dt = 0. \tag{A.76}$$

If $T_* = T_{\min}$ *then* $\Phi_* \le 0$, *if* $T_* = T_{\max}$ *then* $\Phi_* \ge 0$.

A more general formulation of the problem is characteristic for many technical applications: A movement is considered over an infinite time interval, and quality criteria are functionals averaged for infinite time intervals. Periodic controls can be treated as a partial case appearing under a contraction of the class of admissible controls. Such an approach to the control problems for stationary regimes was suggested in [40, 41].

A.5 Maximum Principle for Stochastic Equations with Program Control

The necessary optimality conditions in a form of an analog of Euler-Lagrange equations (5.6), (5.7) hold true for problems without additional constraints for the control and trajectories. This class of problems, though having a partial character, is still of a sufficient importance for applications. The stochastic analog of the maximum principle was obtained for more complex problems with constraints for the control and phase trajectories.

In [138, 195] the relations of the maximum principle for problems of the type (5.1), (5.2) and constraints $u \in U$ are given; more general relations are obtained in [52] for problems with additional constrains in the form of inequalities for extreme points of the phase trajectory. The results of the work [138] are obtained for more weak and easily proved suggestions with respect to a random excitation and control coefficients from (5.1). Let us give without the proof the main theorems of the work [138].

The following optimization problem is considered:

$$\frac{dx}{dt} = f(t,x,u,\omega), \quad u \in U, \quad x(0) = a, \tag{A.77}$$

$$q_1(g_1) = 0_{Z_1}, \quad q_2(g_2) \le 0_{Z_2}, \tag{A.78}$$

$$\Phi(u) = q_3(g_3) + M\int_0^{t_f} \varphi[t, x(t), u(t), \omega]dt = \min, \tag{A.79}$$

where $g_i = Mr_i[x(0), x(t_f), \omega]$. Here $x(t) \in R_n$, $u(t) \in R_m$; piecewise continuous controls $u(t) \in U \subset R_m$ are considered to be admissible; the set U can be infinite; 0_{Z_j} is a zero-vector in a space of the dimension Z_j; the inequality $q_2 \le 0_{Z_2}$ is fulfilled for each component.

It is supposed that $x(0) = a$ is a random vector, and $M|a|^k < \infty$, $k > 1$, ω is an abstract variable characterizing the case.

Let the following conditions be fulfilled:

1) the functions f, φ, $r_i(x_1, x_2, \omega)$, $i = 1, 2$, are continuous and have continuous derivatives with respect to x, x_1, x_2, functions $q_i(g)$ are continuous and have continuous derivatives $q_g = dq_i/dg$, $i = 1, 2, 3$, with probability 1;

2) for all $t \in [0, t_f]$, $x \in R_n$, $u \in U$, functions $f(t,...)$, $\varphi(t,...)$, $r(x_1, x_2, \omega)$ and their derivatives with respect to x (x_1, x_2, respectively) are measurable random processes;

3) for all $t \in [0, t]$ the function $f(t, 0, 0, \omega)$ has finite moments of the order $k > 1$, i.e., $M|f(t, 0, 0, \omega)|^k < \infty$;

4) far all $t \in [0, t_f]$, $u \in U$ there exists such a number A that for each $x \in R_n$ the inequalities

$$|\varphi(t, x, u, \omega)| \le A|x|^k + \zeta_{k_1}(\omega),$$

$$\left|\varphi_x(t,x,u,\omega)\right| \le A|x|^{k-1} + \zeta_{k_1}(\omega)$$

are fulfilled with the probability 1. Here ζ_{kp} are random values, such that $M\left|\zeta_{kp}\right|^k < \infty$. An index k_1 can be found from the equation $1/k_1 + 1/k = 1$.

There exists such a number B that for any x_1, $x_2 \in R_n$, the inequalities

$$\left|r_i(x_1,x_2,\omega)\right| \le B\left(|x_1|^k + |x_2|^k\right) + \zeta_{k_1}(\omega),$$

$$\left|\partial r_i(x_1,x_2,\omega)/\partial x_j\right| \le B\left(|x_1|^{k-1} + |x_2|^{k-1}\right) + \zeta_{k_1}(\omega)$$

hold true with probability 1;

5) the function f satisfies the uniform Lipschitz condition with respect to x for $u \in U$: there exists such a number $L > 0$ that for any x_1, x_2, $t \in [0,t_f]$ the inequality

$$\left|f(t,x_1,u,\omega) - f(t,x_2,u,\omega)\right| \le L|x_1 - x_2|$$

holds true with probability 1. If for any $t \in [0,t_f]$, $x \in R_n$ the random values $|a|$, $|f|$, $|r_i|$ are bounded with probability 1 then the given below Theorems hold true when conditions 1) and 2) are fulfilled.

Let $x_*(t)$, $u_*(t)$ be a solution of the optimal problem, t_* be an optimal time, $a_* = x_*(0)$ be a begin of the optimal trajectory; $\mu_j \in Z_j$, $j = 1,2$ be some vectors, μ_3 be some number. Let us define an absolutely continuous on $[0,t_f]$ function $p(t,\omega)$ and Hamiltonian $H(t,x,u,p)$ by means of relations

$$H(t,x,u,p) = (p, f(t,x,u,\omega)) - \mu_3\varphi(t,x,u,\omega), \tag{A.80}$$

$$\dot p = -H_x(t,x,u,p),$$

$$p(t_f) = -\sum_{i=1}^{3}\left(\mu_i, q_{ig}(g_i)\frac{\partial r_i}{\partial x(t_f)}\right), \tag{A.81}$$

$$p(0) = +\sum_{i=1}^{3}\left(\mu_i, q_{ig}(g_i)\frac{\partial r_i}{\partial x(t_0)}\right).$$

Theorem A.10. *There exist such number μ_3 and vectors $\mu_1 \in Z_1$, $\mu_2 \in Z_2$ that for the optimal trajectory the relations*

$$\sum_{i=1}^{3}|\mu_i| \ne 0, \quad \mu_3 \ge 0, \quad \mu_1 \ge 0_{Z_1}, \quad (\mu_2, q_2) = 0 \tag{A.82}$$

$$MH\big(t_*, x_*(t_*), u_*(t_* - 0), p(t_*)\big) = 0 \tag{A.83}$$

hold true and the following maximum principle

$$MH(t_*, x_*, u_*, p) \geq MH(t, x, u, p) \tag{A.84}$$

holds true for all continuity points $u_*(t)$; $t \in [0, t_f]$ *and all* $u \in U$.

Let us analyze more simple cases.

Theorem A.11. *If the time* t_f *is fixed then the conditions* (A.82), (A.84) *hold true, and all continuity points* $u(t)$ *belong to the interval* $[0, t_f]$.

Theorem A.12. *If there are no constraints* q_1, q_2 *and the time* t_f *is fixed then in* Eqs. (A.81) *it should be considered that* $\mu_3 = 1$ *and*

$$p(t_f) = -\left[q_{3g}(g_3) \frac{\partial r_3}{\partial x(t_f)} \right]. \tag{A.85}$$

The proof of Theorems A.10 – A.12 is based on the abstract theory of the optimal control [139] and it is given in full in [138].

A.6 Main Theorems of the Diffusion Approximation Method

Let us give the construction scheme for an approximated solution for stochastic systems reduced to the standard form with one rapidly rotating phase.

Let us beforehand remind some definitions [45]. Let us consider a probability space $(\Omega, \Im, P)$ where $\Omega = \{\omega\}$ is a space of elementary events, $\Im$ is a σ–algebra of subsets from Ω; *P* is a probability measure on $\Im(P(\Omega) = 1)$. A random process $\xi(t) = \xi(t, \omega)$ at each moment *t* is treated as $\Im$ - measurable value. Denote with $\Im_t$ the smallest σ–algebra with respect to which the random values $\xi(s, \omega)$ are measurable for all $t \leq s$. (If there are no ambiguity, the argument ω can be omitted.)

Let us define the random process $f(t)$ with the following properties [179]. The function $f(t)$ differs from zero only for some finite interval $t \in [0, T]$, is $\Im_t$ – measurable and $\sup_t M|f(t)| < \infty$ (we will denote $f(t) \in M$).

Let us call a function $f(t)$ right-bounded (in mean) if $M|f(t + \delta) - f(t)| \to 0$ for $\delta \to 0$.

Consider an operator $\mathcal{Z}$ with the definition domain $D(\mathcal{Z})$ [174, 179]. We will consider that $f \in D(\tilde{\mathcal{L}})$ and $\tilde{\mathcal{L}}f = g$; if f, $g \in M$ and

$$\lim_{\delta\to\infty} M\left|\delta^{-1}\left[M_t f(t+\delta) - f(t)\right] - g(t)\right| = 0 . \tag{A.86}$$

Here (and below) $M_t f(s) \equiv Mf(s)|F_t$, $t \le s$.

It follows, in particular, from the definition (A.86) [174] that

$$M_t f(t+\delta) - f(t) = \int_t^{t+\delta} M_t \tilde{\mathcal{L}} f(u) du . \tag{A.87}$$

Let $x_\varepsilon(\tau)$ be a solution of the disturbed system

$$dx_\varepsilon / d\tau = F_\varepsilon(x_\varepsilon, \xi_\varepsilon(\tau)), \tag{A.88}$$

where the function $F_\varepsilon(x,\xi)$ is continuous and bounded, and $\xi_\varepsilon(\tau)$ is a right-bounded random process. Let, further, $f(x)$ be a continuously differentiable finite function $\left(f(x) \in \hat{C}_1\right)$ and $f^\varepsilon(\tau) = f(x_\varepsilon(\tau))$. It follows from (A.86) that

$$\tilde{\mathcal{L}} f^\varepsilon(\tau) = \mathcal{L}^\varepsilon f(x_\varepsilon(\tau)) = f_x'(x_\varepsilon(\tau)) F(x_\varepsilon(\tau), \xi_\varepsilon(\tau)). \tag{A.89}$$

In a more general case, when $f^\varepsilon(\tau) = f(\tau, x_\varepsilon(\tau))$,

$$\tilde{\mathcal{L}} f^\varepsilon(\tau) = \tilde{\mathcal{L}}_\tau f(\tau, x_\varepsilon) + f_x'(\tau, x_\varepsilon) F(\tau, x_\varepsilon), \tag{A.90}$$

where $x_\varepsilon = x_\varepsilon(\tau)$ and

$$\tilde{\mathcal{L}}_\tau f(\tau,x) = \lim_{\delta\to\infty} \delta^{-1}\left[M_\tau f(\tau+\delta, x) - f(\tau,x)\right]; \tag{A.91}$$

the equality has the sense of (A.86), x is treated as a fixed parameter (in other words, the operation $\tilde{\mathcal{L}}_\tau$ is analogous to the partial differentiation with respect to τ).

Let us define the diffusion process $x_0(\tau)$ as a solution of the stochastic differential equation

$$dx_0 = b(x_0)d\tau + \sigma(x_0)dw , \quad \sigma(x)\sigma'(x) = A(x). \tag{A.92}$$

Let $\mathcal{L}$ be a generating differential operator of the process $x_0(\tau)$ written in the form (4.11). Then from (4.10) – (4.13) we have

$$M_\tau f(x_0(\tau+\sigma)) - f(x_0(\tau)) = \int_\tau^{\tau+\sigma} M_t \mathcal{L} f(x_0(u)) du . \tag{A.93}$$

Eqs. (A.87) – (A.93) show the computation method for functionals on trajectories of the systems (A.88), (A.92). Let us formulate conditions of closeness for functionals determined on trajectories $x_\varepsilon(\tau)$ and $x_0(\tau)$.

Theorem A.13 [174, 179]. *Let*

1) *there exist a unique solution* $x_0(\tau)$ *of* Eq. (A.92);

2) $f_0(x) \in R_1$ *be a sufficiently smooth function with a compact carrier;*

3) *for each* $T < \infty$ *there exist such function* $f^\varepsilon \in D(\tilde{\mathcal{L}})$ *that for* $0 < \varepsilon \le \varepsilon_0$, $0 \le \tau \le T$

$$\sup_{\tau,\varepsilon} M\left|f^\varepsilon(\tau)\right| < \infty, \tag{A.94}$$

$$\lim_{\varepsilon\to 0} M\left|f^\varepsilon(\tau) - f_0(x_\varepsilon(\tau))\right| = 0, \tag{A.95}$$

$$\lim_{\varepsilon\to 0} M\left|\int_e^T \left[M_\tau \tilde{\mathcal{L}} f^\varepsilon(u) - M_\tau \mathcal{L} f_0(x_\varepsilon(u))\right] du\right| = 0; \tag{A.96}$$

4) *a succession* $x_\varepsilon(\tau)$ *be weakly compact in* $D_n[0,\infty)$ *for* $0 < \varepsilon \le \varepsilon_0$ *and* $x_\varepsilon(0) = x_0(0) = a$.

Then the succession $x_\varepsilon(\tau)$ *weakly converges for* $\varepsilon \to 0$ *to the diffusion process* $x_0(\tau)$, *i.e., a solution of* Eq. (A.92).

As was shown in [179], conditions of Theorem A.13 can be weakened by a replacement of $x_\varepsilon(\tau)$ with a respective truncated process $x_\varepsilon^N(\tau) = x_\varepsilon(\tau)\eta_N$. Here $\eta_N = \{1, |x_\varepsilon(\tau)| \le N; 0, |x_\varepsilon(\tau)| > N\}$. (An assumption on a weak compactness of the process allows a transition to the limit for $N \to \infty$ with a help of standard considerations.)

Let us give a method for a construction of the operator $\mathcal{L}$ in a particular case of the system which has the form

$$\begin{aligned} &\dot{x} = \varepsilon F(\theta, x, \xi(\theta)) + \varepsilon^2 G(x,\theta), \qquad x(0) = a \in R_n, \\ &\dot{\theta} = \omega(x) + \varepsilon H(\theta, x, \xi(\theta)) + \varepsilon^2 D(x,\theta), \\ &\theta(0) = 0, \qquad \theta \in R_1. \end{aligned} \tag{A.97}$$

Here ε is a small parameter, a frequency $\omega(x) \ge \omega_0 > 0$, $\xi(t) \in R_l$ is a random excitation; it is supposed that $MF(\theta, x, \xi(\theta)) = MH(\theta, x, \xi(\theta)) = 0$ for fixed x. All the following transformations are carried out formally; the constraints for the system coefficients for which these transformations hold true [conditions (a) and (b)] are given below.

Let $x(t,\varepsilon) = x_\varepsilon(\tau)$, $\tau = \varepsilon^2 t$, be a solution of the system (A.97). Let us define for the trajectory $x_\varepsilon(\tau)$ some sufficiently smooth function $f^\varepsilon(\tau) = f(x_\varepsilon(\tau))$ and a „truncated" function $f^{\varepsilon^N}(\tau) = f(x_\varepsilon(\tau))\eta_N$. Let us express $f^{\varepsilon^N}(\tau)$ in the form of an expansion

$$f^{\varepsilon^N}(\tau) = \left[f_0(x_\varepsilon) + \varepsilon f_1(\theta_\varepsilon, x_\varepsilon) + \varepsilon^2 f_2(\theta_\varepsilon, x_\varepsilon)\right]\eta_N, \tag{A.98}$$

where $x_\varepsilon = x_\varepsilon(\tau)$, $\theta_\varepsilon = \theta_\varepsilon(\tau)$. Then

$$\tilde{\mathcal{L}} f^{\varepsilon N}(\tau) = \Big[\varepsilon^{-1}\left(f'_{0x}F + \omega\tilde{\mathcal{L}}_\theta f_1\right) + \left(f'_{1x}F + \omega H\tilde{\mathcal{L}}_\theta f_1 + f'_{1x}G + \omega\tilde{\mathcal{L}}_\theta f_2\right)$$
$$+ \varepsilon\left(f'_{2x}F + \omega H\tilde{\mathcal{L}}_\theta f_2 + f'_{1x}G + \omega D\tilde{\mathcal{L}}_\theta f_2\right) + \varepsilon^2\left(f'_{2x}G + \omega D\tilde{\mathcal{L}}_\theta f_2\right)\Big]\eta_N. \tag{A.99}$$

Following [179], let us construct f_1 and f_2 in such a way that secular (with respect to θ) terms are excluded from coefficients by ε and ε^2. Let us write

$$f_1(\theta, x) = \omega^{-1}(x) f'_{0x}(x) \int_0^\infty M_\theta F(u, x, \xi(u))\,du. \tag{A.100}$$

Let us show that $M\left|f'_{0x}F + \omega\tilde{\mathcal{L}}_\theta f_1\right| = 0$. According to the definition,

$$\tilde{\mathcal{L}}_\theta f_1(\theta, x) = \omega^{-1}(x) f'_{0x}(x) \lim_{\delta\to 0_+}\Bigg[M_\theta \int_{\theta+\delta}^\infty M_{\theta+\delta} F(u, x, \xi(u))\,du$$
$$- \int_\theta^\infty M_\theta F(u, x, \xi(u))\,du\Bigg]$$

[here and below the equalities have the sense of (A.86)]. From the properties of the mean value [45] it follows that

$$\tilde{\mathcal{L}}_\theta f_1(\theta, x) = -\omega^{-1}(x) f'_{0x}(x) \lim_{\delta\to 0_+} \int_\theta^{\theta+\delta} M_\theta F(u, x, \xi(u))\,du$$
$$= -\omega^{-1}(x) f'_{0x}(x) F(\theta, x, \xi(\theta)).$$

Thus, coefficients by ε^{-1} in the extension (A.99) turns into zero.

In the analogy with above we can construct

$$f_2(\theta, x) = \omega^{-1}(x)\Bigg\{\int_\theta^\infty \left[M_\theta Q_1(u, x) - MQ_1(u, x)\right]du - \int_0^\theta \left[MQ_1(u, x) - \overline{Q}_1(x)\right]du$$
$$+ f'_{0x}\int_0^\infty \left[M_\theta Q_2(u, x) - MQ_1(u, x)\right]du - f'_{0x}\int_0^\theta \left[MQ_2(u, x) - \overline{Q}_2(x)\right]du\Bigg\} =$$
$$= \omega^{-1}(x)\left[I_1(\theta, x) - S_1(\theta, x) + I_2(\theta, x) - S_2(\theta, x)\right], \tag{A.101}$$

where

$$Q_1(u,x) = f_{1x}'(u,x)F_1(u,x,\xi(u)) = \int_u^\infty M_u\left[f_{0x}'(x)F_1(z,x,\xi(z))\right]_x' dzF(u,x,\xi(u)),$$
$$Q_2(u,x) = -F_1(u,x,\xi(u))H(u,x,\xi(u)) + G(u,x),$$
$$F_1(u,x,\xi) = \omega^{-1}(x)F(u,x,\xi),$$
$$\overline{Q}_j(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} MQ_j(u,x)du$$

(it is supposed that the limit exists uniformly with respect to $t \geq 0$, $x \in S$, S is a bounded domain in R_n).

Accounting for the properties of a conditional mean value [45],

$$\overline{Q}_1(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} d\theta \int_\theta^\infty M\left\{\left[f_{0x}'(x)F_1(u,x,\xi(u))\right]_x F(\theta,x,\xi(0))\right\}du. \tag{A.102}$$

If the function F is differentiable with respect to x, then Eq. (A.102) can be reduced to the form

$$\overline{Q}_1(x) = K'(x)\frac{\partial f_0}{\partial x} + \frac{1}{2}\mathrm{Tr}\, A(x)\frac{\partial^2 f_0}{\partial x^2}, \tag{A.103}$$

where

$$K(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} d\theta \int_\theta^\infty K(u,\theta,x)du,$$
$$A(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T+t_0} d\theta \int_\theta^\infty A(u,\theta,x)du, \qquad A = \sigma\sigma', \tag{A.104}$$
$$K(u,\theta,x) = M\left[F_{1x}(u,x,\xi(u))F(\theta,x,\xi(\theta))\right],$$
$$A(u,\theta,x) = M\left[F_1(u,x,\xi(u))F'(\theta,x,\xi(\theta))\right].$$

Substituting (A.100) into (A.101), we will get

$$\tilde{\mathcal{L}}^\varepsilon f^{\varepsilon N}(\tau) = \left[\left(\mathcal{L} + \varepsilon R_1 + \varepsilon^2 R_2\right)f_0(x_\varepsilon)\right]\eta_N, \tag{A.105}$$

where

$$\mathcal{L} = b'(x)\frac{\partial}{\partial x} + \frac{1}{2}\mathrm{Tr}\, A(x)\frac{\partial^2}{\partial^2 x}, \tag{A.106}$$
$$b(x) = K(x) + \overline{Q}_2(x),$$

R_1, R_2 are operators corresponding to coefficients by ε, ε^2 in Eq. (A.99).

The function f_2 is constructed in such a way that the components depending on θ in a coefficient by ε^0 turn into zero; a proof is analogous to the estimation of the coefficient by ε^{-1}. Conditions of Theorem A.13 are fulfilled when

$$M\left|f_j(\theta,x)\right|<\infty, \quad M\left|R_j f_0(\theta,x)\right|<\infty, \quad j=1,2, \tag{A.107}$$

for $|x|\le N$. Let us formulate requirements to coefficients of the system providing the fulfillment of the conditions (A:107).

Let the following conditions hold true:

(A) the functions F, H can be presented in the form

$$F(\theta,x,\xi(\theta))=F_0(\theta,x)\xi(\theta),$$
$$H(\theta,x,\xi(\theta))=H_0(\theta,x)\xi(\theta),$$

random disturbances $\xi(\theta)$ belong to one of two types:

a) $\xi(\theta)$ is a stationary right-continuous normal Markov random process;

b) $\xi(\theta)$ is a bounded with probability 1 $(|\xi(\theta)|<k)$ stationary process satisfying the condition of the uniformly strong mixing [27].

(B) coefficients $U=(F_0,H_0,G,D)$ and the frequency $\omega(x)$ satisfy the conditions:

1) the functions U are measurable with respect to θ for $\theta\ge 0$ uniformly with respect to $x\in S$ in any bounded domain $S\subset R_n$;

2) the functions U, ω are continuous with respect to x for all $x\in R_n$ uniformly with respect to $\theta\ge 0$;

3)derivatives of the functions F_0, H_0, ω and the second derivatives of F_0, ω with respect to x are continuous for $x\in R_n$ and bounded for $x\in S$ uniformly with respect to $\theta\ge 0$;

4) the limits (A.104) exist uniformly with respect to $x\in S$, $t_0\ge 0$.

Let us specify conditions (A). Suppose for the sake of simplicity that $\xi(\theta)$ is a scalar process with a correlation function $K(z)$. Then the following conclusions hold true.

a) for a normal Markov process

$$M_\theta\xi(u)=M\xi(u)|\xi(\theta)=\chi(\theta-u)\xi(\theta), \quad \theta\le u,$$
$$M_\theta\xi^2(u)-K(0)=\chi^2(\theta-u)(\xi^2(\theta)-K(0)),$$
$$M_\theta\xi(u_1)\xi(u_2)-K(u_2-u_1)=\chi(u_2-u_1)M(\xi^2(u_1)-K(0)),$$
$$\theta\le u_1\le u_2.$$

Here $\chi(t)=K(t)/K(0)$. Accounting for the properties of higher moments of the normal process, it is easy to get an estimate

$$M\left|M_\theta\left[\xi(u_1)\right]^0\xi(u)\right]^0\ldots\xi(u_n)\right]^0\right| \le \sum_{j=1}^{n} c_j \left|\chi^{m_{1j}}(u_1-\theta)\ldots\chi^{m_{nj}}(u_n-u_{n-1})\right|, \quad \text{(A.108)}$$

$$\theta \le u_1 \le \ldots \le u_n,$$

where $m_{kj} \in [1, n]$, $k = 1, \ldots, n$, constants c_j depend on values $M|\xi(\theta)|^j$. A symbol $[\ ]^0$ denotes the centering operation for a random value: $|\zeta|^0 = \zeta - M\zeta$. Analogous estimates remain for vector random processes.

Accounting for exponential type of decrease for the functions $|\chi(t)|$ for Markov processes we can write

$$\int_\theta^\infty |\chi^m(t)|dt < \infty, \qquad m > 0. \quad \text{(A.109)}$$

Using (A.108), (A.109), it is easy to obtain that for normal disturbances and under the fulfillment of conditions (B) for $x \in S$, $\theta \ge 0$

$$M|f_1(\theta,x)| < \infty, \qquad M|I_j(\theta,x)| < \infty, \qquad j = 1, 2. \quad \text{(A.110)}$$

b) For processes satisfying the condition of the uniformly strong mixing, the estimates [27]

$$M|M_\theta\xi(u)| \le c_1\alpha(u-\theta), \qquad \theta \le u,$$

$$M|M_\theta\xi(u_1)\xi(u_2) - K(u_2-u_1)| = c_2\alpha^{1/2}(u_1-\theta)\alpha^{1/2}(u_2-u_1),$$

$$\theta \le u_1 \le u_2, \quad \text{(A.111)}$$

$$M\left|M_\theta\left[\ldots\left[\left[\xi(u_1)\right]^0\xi(u_2)\right]^0\ldots\xi(u_n)\right]^0\right| \le c_n\alpha^\gamma(u_1-\theta)\ldots\alpha^\gamma(u_n-u_{n-1}),$$

$$\theta \le u_1 \le \ldots \le u_n,$$

hold true, where the function $\alpha(\theta)$, which diminishes for $\theta \to \infty$, is called a *coefficient of uniformly strong mixing*, $\gamma = 1/(n-1)$, the constants c_n depend on $M|\xi(\theta)|^n \le k^n$. It is supposed that $\alpha(\theta)$ satisfies a condition

$$\int_0^\infty \alpha^\gamma(\theta)d\theta. \quad \text{(A.112)}$$

At the estimation of terms in (A.100) only the moments of the order not higher than 3 are left, so the condition (A:112) should be satisfied for $\gamma = 1/2$.

It is obvious that the estimates (A.110) hold true for our assumptions.

Let us estimate the deterministic terms S_1, S_2.

Consider $MQ_j(u,x) = MQ_j(\sigma/\varepsilon^2, x)$ and suppose that the functions $MQ_j(\sigma/\varepsilon^2, x)$ are uniformly integrally continuous for $\varepsilon \to 0$, i.e., that there exist limits

$$\overline{Q}_j(x) = \lim_{\varepsilon \to 0} \frac{1}{\varphi_0} \int_0^{\varphi_0} MQ_j(\sigma/\varepsilon^2, x) d\sigma \tag{A.113}$$

uniformly with respect to $x \in K$, $\varphi_0 > 0$. If the functions $Q_j(\theta, x)$ are periodic or uniformly quasi-periodic with respect to θ then it can be easily shown that for all $\varphi_0 > 0$

$$\left| \int_0^{\varphi_0} \left[MQ_j(\sigma/\varepsilon^2, x) - \overline{Q}_j(x) \right] d\sigma \right| \leq c\varepsilon^2, \qquad c = \text{const}. \tag{A.114}$$

It follows form (A.113), (A.114) that

$$|S_j(\theta, x)| < \infty, \qquad j = 1, 2, \tag{A.115}$$

uniformly with respect to $x \in S$, $\theta \geq 0$. Thus , in the given domain of the variables

$$M|f_j(\theta, x)| < \infty, \qquad j = 1, 2. \tag{A.116}$$

In analogy with the above considerations it is easy to obtain

$$M\left| \int_\theta^\infty M_\theta R_j(u,x) du \right| \leq \infty. \tag{A.117}$$

It follows from (A.115), (A.116) that the requirements (A.94) – (A.96) of Theorem A.13 hold true. The weak compactness of the succession $x_\varepsilon(\tau)$ and the existence of the solution $x_0(\tau)$ for the finite interval $\tau \in [0, T]$ are proved in the same way as in [179].

Thus, the following Theorem holds true.

Theorem A.14. *Let*

1) *functions F, G, H, D satisfy conditions* (A), (B);

2) *the solution of the system* (A.92) *corresponding to the operator* (A.106) *exist and be unique for any initial conditions* $x_0(\tau) = x_\varepsilon(\tau) = a \in S$.

Then on the interval $0 \leq \tau \leq T$ *the process* $x_\varepsilon(\tau)$ *– the solution of* Eq. (A.101) *– weakly converges for* $\varepsilon \to 0$ *to a continuous with the probability 1 Markov process* $x_0(\tau)$ *– the solution of* Eq. (A.92). *The coefficients* $b(x)$ *and* $A = \sigma(x)\sigma'(x)$ *are calculated with the use of* Eqs. (A.104). (A.105).

The process $x_0(\tau)$ will be called a *limit Markov (diffusion) process* and Eq. (A.92) will be called a *limit stochastic (diffusion) equation.*

Theorem A.14 serves as a generalization of the Theorem 5 from Chapter 4 in [179, p. 83] for a case of systems with a rapidly rotating phase.

Remark 1. Theorems A.13, A.14 give a method for an approximated calculations of the functionals for the trajectories of the disturbed system (A.97). In a particular case of the functionals of the form $V_\varepsilon = M_{0,\varepsilon}\varphi\left(x_\varepsilon\left(\tau_f\right)\right)$ it is possible to construct a direct estimate of the closeness of V_ε to $V_0 = M_{0,\varepsilon}\varphi\left(x_\varepsilon\left(\tau_f\right)\right)$. Let us prove that

$$\left|V_\varepsilon - V_0\right| \le C\varepsilon, \tag{A.118}$$

where C is a constant which does not depend on ε, $0 \le \tau_f \le T$, $\varphi \in R_1$.

It follows from conditions (B) that coefficients of the operator $\mathcal{L}$ are continuous and continuously differentiable with respect to x in any bonded domain $|x| < N$. Let us additionally suppose that the operator $\mathcal{L}$ is uniformly parabolic [95] in the domain under study and satisfies the regularity conditions [129]. Then there exists a unique solution $x_0(\tau)$ of the limit equation (A.92) determined for all $\tau \ge 0$ (with probability 1) [129]. In other words, the diffusion process $x_0(\tau)$ is regular, i.e., its trajectory does not leave any bounded domain in a finite time.

Let us construct the function $f(\tau, x)$ as a solution of the Cauchy problem

$$\frac{\partial f}{\partial \tau} + \mathcal{L}f = 0, \qquad f\left(\tau_f, x\right) = \varphi(x). \tag{A.119}$$

Let $\varphi(x) \in \hat{C}_4$ be a function with a compact carrier determined for a set $|x| < N$. If the operator $\mathcal{L}$ possesses the above mentioned properties then the solution of the problem (A.119) exists and $f(t,x) \in \hat{C}_{2,4}$ in the given domain. And $f(\tau, x) = M_{0,\varepsilon}\varphi\left(x_\varepsilon\left(\tau_f\right)\right)$ (Section 4.1).

If the process $x_\varepsilon(\tau) \in D_n[0,\infty)$ then there exists such value τ_M that $f(\tau, x_\varepsilon(\tau)) \in M$ for $0 \le \tau \le \tau_f \le \tau_M$. Then, accounting for (A.87), (A.98), (A.105) and also considering the dependence of f on τ, we can write

$$M_{\tau,x} f\left(\tau_f, x_\varepsilon\left(\tau_f\right)\right) - f(\tau, x) + \varepsilon M_{\tau,y} f\left(\tau_f, y_\varepsilon\left(\tau_f\right), \varepsilon\right) =$$

$$= \int_\tau^{\tau_f} M_{\tau,x}\left[f_u\left(u, x_\varepsilon(u)\right) + \mathcal{L}f\left(u, x_\varepsilon(u)\right)\right]du + \varepsilon \int_\tau^{\tau_f} M_{\tau,y}\rho\left(u, y_\varepsilon(u), \varepsilon\right)du. \tag{A.120}$$

The following notation is used in (A.120): $x_\varepsilon(\tau) = x$, $\theta_\varepsilon(\tau) = \theta$, $(\theta, x) = y$,

$(\theta_\varepsilon, x_\varepsilon) = y \in R_{n+1}$, F and ρ are respective rest terms in (A.98), (A.105).

It follows for (a.119) that the first term in the right-hand part of (A.120) becomes zero. Thus, accounting for (A.107), we can write

$$\left| M_{\tau,x} f\left(\tau_f, x_\varepsilon(\tau_f)\right) - f(\tau, x) \right| \le \varepsilon \left[C_1 + C_2 (\tau_f - \tau) \right], \tag{A.121}$$

where $C_1 > 0$, $C_2 > 0$ are constants that do not depend on ε. On the other hand, we have form (A.119) $f\left(\tau_f, x_\varepsilon(\tau_f)\right) = \varphi\left(x_\varepsilon(\tau_f)\right)$. Thus,

$$\left| M_{\tau,x} \varphi\left(x_\varepsilon(\tau_f)\right) - M_{\tau,x} \varphi\left(x_0(\tau_f)\right) \right| \le \varepsilon \left[C_1 + C_2 (\tau_f - \tau) \right]. \tag{A.122}$$

Considering $\tau = 0$, $x = a$, we get

$$\left| V_\varepsilon - V_0 \right| \le \varepsilon \left[C_1 + C_2 \tau_f \right], \tag{A.123}$$

i. e, the estimate (A.118) holds true.

Strictly speaking, all the transformations hold true for „truncated“ processes in the domain $|x_\varepsilon(\tau)| < N$. Using the Chebyshev inequality [45], it can be easily shown that from the regularity of $x_0(\tau)$ and estimates (A.122), (A.123) follows the regularity of the process $x_\varepsilon(\tau)$ (the proof can be constructed in the same way as in Theorem 4.1 from Section 3 in [129]. Thus, the estimate (A.123) remains true for all finite values of τ_f).

The estimate (A.123) was obtained in [153, 190] for systems in the standard form for more strict constraints for the system coefficients).

Remark 2. The system in the standard form can be treated – as for a deterministic case – as a particular case of systems with a rapidly rotating phase

$$\dot{x} = \varepsilon F(t, x, \xi(t)) + \varepsilon^2 G(t, x), \qquad \dot{t} = 1. \tag{A.124}$$

All the statements of Theorem A.14 remain true. Coefficients of the limit diffusion equation (A.92) can be still calculated with the use of Eqs. (A.101) – (A.104), (A.106) (for $\theta = t$). Here $Q_2 \equiv G(t, x)$ and Eqs. (A.101) – (A.104), (A.106) coincide with relations from [132]. In particular, if the components F_{ij} of the matrix F_0 and G_i of the vector G are periodic with respect to t with a period T_0, $\xi(t)$ is an n-dimensional stationary random process with a zero mean and a correlation matrix $K(u)$, then coefficients b_i, a_{ij} are calculated in accordance with relations from [132]

$$G_i(x) = \frac{1}{T_0} \int_0^{T_0} G_i(t, x) dt, \tag{A.125}$$

$$K_i(x) = \sum_{j,r,m=1}^{n} \frac{1}{T}\int_{0}^{T_0} ds \int_{-\infty}^{0} \frac{\partial F_{ir}(s,x)}{\partial x_j} F_{jm}(s+u,x)K_{rm}(u)du\,,$$

$$a_{ij}(x) = \sum_{r,m=1}^{n} \frac{1}{T_0}\int_{0}^{T_0} ds \int_{-\infty}^{\infty} F_{ir}(s,x)F_{jm}(t,x)K_{rm}(t-s)dt\,,$$

$$b_j(x) = G_i(x) + K_i(x), \qquad A(x) = \{a_{ij}(x)\}_{i,j=1}^{n} = \sigma(x)\sigma'(x). \tag{A.126}$$

Here $K_{rm}(u) = M[\xi_r(t+u)\xi_m(t)]$ are components of a correlation matrix $K(u)$.

If the coefficients F_0, G depend on the „slow" time τ, $F_0 = F_0(t,\tau,x)$, $G_0 = G_0(t,\tau,x)$, then the variable τ can be determined from the equation $\dot{\tau} = \varepsilon^2$. And coefficients of the limit diffusion equation (4.31) are determined by Eqs. (4.32) – (4.34).

Remark 3. It can be easily shown that the stochastic variant of Theorem A.4 on the partial averaging holds true.

Let the coefficients F, G of Eq. (A.119) satisfy the conditions of Theorem. Then the solution $x_\varepsilon(\tau)$ of Eq. (A.119) weakly converges on the interval $0 \le \tau \le T$ (for $\varepsilon \to 0$) to the solution $x_{0\varepsilon}(\tau)$ of the stochastic differential equation

$$dx_{0\varepsilon} = \left[K(x_{0\varepsilon}) + G(\tau/\varepsilon^2, x_{0\varepsilon})\right]d\tau + \sigma(x_{0\varepsilon})dw\,, \tag{A.127}$$

where coefficients K, σ are calculated with the use of Eqs. (A.104). And the estimate (A.118) remains true. It is obvious that in the case of the partial averaging only the variables depending on the random disturbance are transformed.

References

1. Avetisyan VV, Akulenko LD, Bolotnik NN (1986) Modeling and optimization of transport motions of industrial robot (in Russian). Izv. AN SSSR. Tekhn. Kibern. 3
2. Avetisyan VV, Akulenko LD, Bolotnik NN (1986) Optimization of control regimes for manipulation robots with account for energy losses (in Russian). Izv. AN SSSR. Tekhn. Kibern. 3
3. Isaacs R (1965) Differential games. Wiley, New York
4. Akulenko LD (1976) Approximate solution for problems of optimal periodic control by means of canonical separation of motions (in Russian). Prikl. Mat. Mekh. 40
5. Akulenko LD (1980) Asymptotic solution of two-point boundary–value problems (in Russian). Prikl. Mat. Mekh. 44:633
6. Akulenko LD (1994) Problems and methods of optimal control. Kluwer Academic Publishers, Dordrecht e. a. [Russian original: Moscow, 1987].
7. Akulenko LD, Bolotnik NN (1986) Optimal control synthesis for transport motions of robot manipulators (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 4:21
8. Akulenko LD, Kolmanovskii VB (1975). On one model problem of motion control for a solid in atmosphere (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 2:15
9. Akulenko LD, Tchernousko FL (1975) Averaging method in optimal control problems (in Russian). Zh. Vytch. Mat. Mat. Fiz. 15: 869
10. Anisovitch VV (1982) On optimization of non-linear nearly periodic oscillatory processes (in Russian). Avtom. Telemekh. 3:190
11. Anisovitch VV, Kryukov BI (1981) On optimization of nearly periodic oscillations (in Russian). Avtom. Telemekh. 12:168
12. Aslanyan AA (1982) Optimality conditions in control problems for systems with impulse excitations (in Russian). DAN SSSR. Ser. A 11:3
13. Aslanyan AA (1982) Averaging of motion equations in control problems for systems of differential equations with impulse excitation. In: Theory of optimal solutions. Kiev, pp 57-62
14. Astashev VK, Babitsky VI (1982) Tuning methods for resonant machines (in Russian). Mashinoved. 5:3
15. Ashavskii AM, Volpert AYa, Sheinbaum VS (1978) Power impulse systems (in Russian). Mashinostroenie, Moscow.
16. Babitsky VI (1998) Theory of vibroimpact systems with applications Springer, Berlin e. a.
17. Babitsky VI, Kovaleva AS (1986) Optimal control in resonant systems of manipulators (in Russian). Mashinoved. 2:21
18. Babitsky VI, Kovaleva AS (1987) Control for resonant manipulator with minimization of energy losses (in Russian). Izv. AN SSSR, Tekhn. Kibern. 3: 108
19. Babitsky VI, Kovaleva AS (1988) Optimal periodic control for asymmetric vibroimpact systems (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 1:50
20. Babitsky VI, Kovaleva AS, Krupenin VL (1982) Study of quasi-conservative vibroimpact systems by means of averaging method (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 1:41
21. Babitsky VI, Kolovsky MZ (1967) Oscillation of a linear system with limiters under

random excitations (in Russian). Inzh. Zh. Mekh. Tverd. Tela 3:147.
22. Babitsky VI, Kolovsky MZ (1970) To the theory of vibroimpact systems (in Russian). Mashinoved. 1:25
23. Babitsky VI, Kolovsky MZ (1976) To the analysis of resonant regimes of vibroimpact systems (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela. 4:88
24. Babitsky VI, Krupenin VL (1985). Oscillations in strongly non-linear systems (in Russian). Nauka, Moscow
25. Bart ME, Kolovsky MZ On optimization of parameters of elastic coupling and a motion control of machine units (in Russian). Mashinoved. 1:12
26. Bakhvalov N, Panasenko G. (1993) Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials. Kluwer, Dordrecht e. a.. [Russian original: Moscow 1984]
27. Billingsley P (1968) Convergence of probability measures. John Wiley, New York
28. Bolotin VV (1984) Random vibrations of elastic systems. Nijhoff, Boston [Russian original: Moscow, 1979]
29. Bolotnik NN (1983) Optimization of amortization systems (in Russian). Nauka, Moscow
30. Borodin AN (1977) Limit theorem for solution of differential equations with a random right-hand part (in Russian). Teor. Verojatn. Primen. 22:498
31. Bryson AE Jr, Ho YC (1975) Applied optimal control. Hemisphere Publishing, Washington, DC
32. Brodsky YaS, On higher approximations of the averaging method to stochastic differential equations (in Russian). Teor. Slutch. Proz. 5:9
33. Burd VSh (1986) On applicability of the averaging method for finite and infinite time intervals in systems with impulses (in Russian). Prikl. Mat. Mekh. 1:50
34. Burd VSh, Matveev VM (1985). Asymptotic methods for infinite interval in problems of non-linear mechanics (in Russian). Jaroslavl
35. Butkovskij AG (1969) Distributed control systems. Elsevier, New York [Russian original: Moscow, 1965]
36. Veitz VL, Kolovsky MZ, Kotchura AE (1984) Dynamics of guided machine units (in Russian). Nauka, Moscow
37. Freidlin MI, Wentzell AD (1984) Random perturbations of dynamical systems. Springer New York e. a. [Russian original: Moscow, 1979].
38. Volosov VM, Morgunov BI (1971) Averaging method in theory of non-linear oscillatory systems (in Russian). Moscow State University Press, Moscow
39. Vonam VM (1971) Optimal stationary control of linear systems under excitation by the state–depending noise (in Russian). In: Non-linear and optimal systems. Nauka, Moscow
40. Gaitsgori VG (1985) Asymptotic properties of control problems on infinite time intervals (in Russian). Izv. AN SSSR. Tekhn. Kibern. 3:161
41. Gaitsgori VG (1986) To the problem of utilization of the averaging method in control problems (in Russian). Differ. Uravn. 22:22
42. Gantmacher, FR (1975) Lectures on analytical mechanics. Mir Publishers, Moscow [Russian original: Moscow, 1966]
43. Gel'fand IM, Shilov GE (1963) Generalized functions. Academic Press, New York: [Russian original: Moscow, 1958].
44. Gertz ME (1984) Limit possibilities and effectiveness of resonant machines (in Russian). Mashinoved. 5:15
45. Gichman II, Skorochod AV (1969) Introduction to the theory of random processes Saunders, Philadelphia [Russian original: Moscow, 1965].
46. Gichman II, Skorochod AV (1972) Stochastic differential equations. Springer, Berlin e. a. [Russian original: Moscow, 1968]
47. Dimentberg MF (1988) Statistical dynamics of nonlinear and time-varying systems.

Research Studies Press, Taunton [Russian original: Moscow, 1980]

48. Dimentberg MF (1982) Method of moments in dynamic problems of systems with randomly changing parameters (in Russian). Prikl. Mat. Mekh. 2:218
49. Dimentberg MF, Isikov NE (1983) Combination resonances in systems with periodic parametric and random external excitations (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 4:63
50. Dimentberg MF, Menyailov AI (1976) Some stochastic problems of dynamics of vibroimpact systems (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 4:63.
51. Dimentberg ME, Menyailov AI (1978) Random oscillations of one-mass vibroimpact systems (in Russian). Dokl. AN SSSR. 238:285
52. Dokutchaev HG, Yakubovich VA (1984) Optimal program control of stochastic object with state constraints at each moment of time (in Russian). Avtom. i Telemekh. 7:49
53. Doob JL (1953) Stochastic processes. Wiley, New York e. a.
54. Dyachkovsky SV, Khokhel OS (1983) On one variant of central limit theorem for adaptive functionals of Markov processes in a scheme of asymptotic phase enlarging (in Russian). In: Application of analytic methods in probability theory. Kiev, pp. 62-72.
55. Zhikov VV (1973) Averaging principle for parabolic equations with varying main term (in Russian). Dokl. AN SSSR, 208:302
56. Zhikov VV (1976) Some applicability problems of dichotomy: Averaging principle (in Russian). Izv. AN SSSR. Ser. Mat. 40:1380
57. Zhuravlev VF (1976) Method of analysis of vibroimpact systems with the use of special functions (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 2:30
58. Zhuravlev VF, Menyailov AI (1978) A study of vibroimpact system with bounded excitation (in Russian). Izv AN SSSR. Mekh. Tverd. Tela 2:45
59. Zhurbenko IG (1987) An analysis of stationary and homogeneous random systems (in Russian). Moscow State University Press, Moscow
60. Zadeh LA, Desoer CA (1963) Linear system theory: The state space approach. McGraw-Hill, New York
61. Zakharov MG, Kultchitsky OYu (1983) A stability study of systems of linear differential equations with parametric excitations by correlated noise by means of integro-differential inequalities (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 1:30
62. Zevin AA (1976) An optimal regime in vibroimpact systems with one degree of freedom (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 2:35
63. Zevin AA (1980) To the problem of optimal control for periodic processes (in Russian). Avtom. i Telemekh. 3:20
64. Zuev LA (1980) Program control for some partially observable stochastic systems (in Russian). In: The theory of random process. Kiev, pp. 65-68
65. Ibragimov IA, Linnik YuV (1971) Independent and stationary sequences of random variables. Wolters-Noordhoff, Groningen [Russian original: Moscow, 1965]
66. Ibragimov IA, Rozanov YuA (1978) Gaussian random processes. Springer, New York [Russian original: Moscow, 1970]
67. Izrajlovitch MYa (1969) An optimal law of periodic motion of one-mass vibroimpact system (in Russian). Mashinoved. 1
68. Il'in AM, Has'minskii RZ (1963) An asymptotic behavior of solutions of parabolic equations and an ergodic property of non-homogeneous diffusion processes (in Russian). Mat. Sb. 60(3):366
69. Kwakernaak H, Sivan R (1972) Linear optimal control systems. Wiley-Interscience, New York
70. Kecs W, Teodorescu PP (1974) Applications of the theory of distributions in mechanics. Abacus Press, Tunbridge Wells
71. Kobrinsky AA, Kobrinsky AE (1983) Two-dimensional vibroimpact systems (in Russian). Nauka, Moscow

72. Kobrinsky AA, Kobrinsky AE (1973) Vibroimpact systems (in Russian). Nauka, Moscow
73. Kovaleva AS (1982) An optimal control of periodic oscillations of a vibroimpact system (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 6:40
74. Kovaleva AS (1983) A study of vibroimpact resonances in systems with random parametric excitations (in Russian).Mashinoved. 3:23
75. Kovaleva AS (1983) An optimization of periodic motions of some vibroimpact systems (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 6:35
76. Kovaleva AS (1984) A periodic control for some mechanic systems (in Russian). Mashinoved. 3:23
77. Kovaleva AS (1984) Program control for weak-control oscillatory systems (in Russian). Izv. AN SSSR. Tekhn. Kibern. 4:81
78. Kovaleva AS (1984) Applications of asymptotic methods to some stochastic dynamic problems of vibroimpact systems (in Russian).Prikl. Mat. Mekh. 5:733
79 Kovaleva AS (1986) On optimal control for wide-band random excitation (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 3:72
80. Kolmanovsky VB (1980) Quasi-optimal program control for some stochastic systems (in Russian). Differ. Uravn. 26:795
81. Kolmanovsky VB (1975) Application of excitation method to some problems of optimal control (in Russian). Prikl. Mat. Mekh. 39:788
82. Kolovsky MZ (1966) Non-linear theory of vibro-protective systems (in Russian). Nauka, Moscow
83. Kolovsky MZ (1976) Automatic control of vibro-protective systems (in Russian). Nauka, Moscow
84. Kolovsky MZ (1960) On existence conditions of periodic solutions of the system of differential equations with discontinuous right-hand parts (in Russian). Prikl. Mat. Mekh. 4:738
85. Kolovsky MZ (1963) On application of the small-parameter method for determination of discontinuous periodic solutions (in Russian). In: Proceedings of Intern. Symposium on non-linear oscillations (Vol. 1) Kiev, pp. 264-276
86. Kolovsky MZ (1964) On choice of the distribution law for linearization of non-linear differential equations (in Russian). Trudy LPI 235
87. Kolovsky MZ (1966) On accuracy estimation for solutions obtained by method of statistic linearization (in Russian). Avtom. i Telemekh. 10:43
88. Kolovsky MZ (1971) On accuracy estimation for approximated stochastic solutions of non-linear differential equations (in Russian). In: Non-linear and optimal systems. Nauka, Moscow, pp. 235-245
89. Kolovsky MZ (1977) On optimization of active vibro-protective systems (in Russian). Mashinoved. 5:42
90. Kolosov GE (1984) Synthesis of optimal automatic under random excitations (in Russian). Nauka, Moscow
91. Krasnoselsky MA, Krein SG (1955) On averaging principle in non-linear mechanics (in Russian). Usp. Mat. Nauk 10:147
92. Kruzhkov SN, Kamynin VL (1983) Convergence of solutions of quasi-linear parabolic equations with weakly convergible coefficients (in Russian). Dokl. AN SSSR. 270:533
93. Kruzhkov SN, Kamynin VL (1985) On limit transition in quasi-linear parabolic equations (in Russian). Trudy MIAN. 167:183
94. Krylov NV (1980) Controlled diffusion processes. Springer, New York [Russian original: Moscow, 1977]
95. Ladyzenskaja OA, Solonnikov VA, Ural'ceva NN (1968) Linear and quasi-linear equations of parabolic type. American Mathematical Society, Rhode Island, Providence [Russian original: 1967]
96. Landa PS (1996) Nonlinear oscillations and waves in dynamical systems. Kluwer,

Dordrecht e. a.

97. Larin PS (1980) Self-sustained oscillations in systems with finite number of degrees of freedom (in Russian). Nauka, Moscow
98. Levenshtam VB (1976) Averaging method and bifurcation of bounded solutions of abstract parabolic equations (in Russian). Sib. Mat. Zh. 17:1058
99. Lurie AI (1951) Operational calculus (in Russian). Gostekhizdat, Moscow
100. Malkin JG (1959) Theorie der Stabilität einer Bewegung. Oldenbourg, München
101. Mirkina AS (1977) Determination of the second instability domain for equations with random coefficients (in Russian). Izv. AN SSSR. Mekh. Tverd. Tela 6:95
102. Mitropolsky YuA (1971) Averaging method in non-linear mechanics (in Russian). Naukova Dumka, Kiev
103. Moiseev NN (1969) Asymptotic methods of non-linear mechanics (in Russian). Nauka, Moscow
104. Morozov SF, Smirnov IP (1979) On one control problem for stochastic process (in Russian). Differ. Uravn. 15:121
105. Nayfeh AH (1973) Perturbation methods. Wiley, New York
106. Neimark YuI (1972) Method of point mapping in the theory of non-linear oscillations (in Russian). Nauka, Moscow
107. Palmov V (1998) Vibrations of elasto-plastic bodies. Springer, Berlin e. a. [Russian original: Moscow, 1976]
108. Panovko YaG (1977) Introduction into the theory of mechanical impact (in Russian). Nauka, Moscow
109. Pervozvanskii AA (1965) Random processes in nonlinear control systems. Academic Press, New York e .a. [Russian original: Moscow, 1962]
110. Pervozvanskii AA, Gaitsgori VG (1988) Theory of suboptimal decisions: decomposition and aggregation. Kluwer, Dordrecht e. a. [Russian original: Moscow, 1979]
111. Plotnikov VA (1976) Asymptotic methods in optimal control problems (in Russian). Odessa State University Press, Odessa
112. Plotnikov VA (1978) A method of partial averaging in problems of terminal control (in Russian). Differ. Uravn. 14:376
113. Plotnikov VA, Zverkova TS (1978) Averaging of boundary-value problems in terminal problems of optimal control (in Russian). Differ. Uravn. 14:1381
114. Pontryagin LS (1965) The mathematical theory of optimal processes. Wiley, New York
115. Rosenwasser EN (1969) Oscillations of non-linear systems (in Russian). Nauka, Moscow
116. Rosenwasser EN (1973) Periodic non-stationary control systems (in Russian). Nauka, Moscow.
117. Samoilenko AM, Perestyuk NA (1995) Impulsive differential equations. World Scientific, Singapore e. a. [Russian original: Kiev, 1987]
118. Svetlitsky VA (1976) Random oscillations of mechanical systems (in Russian). Mashinostroenie, Moscow
119. Simonenko IB (1970) Substantiation of averaging method for abstract parabolic equations (in Russian). Mat. Sb. 81:53
120. Stratonovic RL (1967) Topics in the theory of random noise. Gordon and Breach, New York [Russian original: Moscow, 1961]
121. Stratonovic RL (1968) Conditional Markov processes and their application to the theory of optimal control. American Elsevier Publishing Company, New York [Russian original: Moscow, 1966]
122. Stratonovic RL (1969) Maximum principle and statistical problems of optimal control (in Russian). In: Optimal systems of automatic control. Nauka, Moscow
123. Tonkov EL (1977) Optimal periodic motions of guided system (in Russian). Mat. Fiz. 21:45

124. Tonkov EL (1977) Optimal periodic control (in Russian). Mat. Fiz. 22:54
125. Troitsky VA (1976) Optimal processes of oscillations of mechanical systems (in Russian). Mashinostroenie, Leningrad
126. Turbin AF, Khokhel OS (1982) Limit theorems for dynamic systems with coefficients depending on Markov processes (in Russian). In: Applied problems of probability theory. Kiev, pp. 117-133
127. Fleming, WH, Rishel R (1975) Deterministic and stochastic optimal control. Springer-Verlag, Berlin
128. Frolov KV, Furman FA (1990) Applied theory of vibration isolation system. Hemisphere Publishers, New York e. a. [Russian original: Moscow, 1980]
129. Has'minskii RZ (1980) Stochastic stability of differential equations. Sijthoff & Noordhoff, Alphen aan den Rijn, [Russian original: Moscow, 1969]
130. Has'minskii RZ (1963) On the averaging principle for parabolic and elliptic equations and Markov processes with small diffusion (in Russian). Teor. Verojatn. Primenen. 8:3
131. Has'minskii RZ (1966) On random processes determined by differential equations with a small parameter (in Russian). Teor. Verojatn. Primenen. 11:240
132. Has'minskii RZ (1966) Limit theorem for solution of differential equations with a random right-hand part (in Russian). Teor. Verojatn. Primenen. 11:444
133. Tzirlin AM (1986) Optimal control of technological processes (in Russian). Energoatomizdat, Moscow
134. Tchernousko FL, Akulenko LD, Sokolov BN (1976) Periodic control (in Russian). Nauka, Moscow
135. Tchernousko FL, Kolmanovsky VB (1977) Numerical and approximate methods of optimal control (in Russian). In: Achievements in science and technology. Mathematical analysis (Vol. 14). VINITI Publishers, Moscow, pp.101-166
136. Tchernousko FL, Kolmanovsky VB (1978) Optimal control for random excitations (in Russian). Nauka, Moscow
137. Schwartz L. (1966) Mathematics for the physical sciences. Hermann, Paris
138. Yakubovich VA, Dokutchaev NG (1982) A maximum principle for stochastic differential equations with deterministic control (in Russian). Kibern Vytch Tekhn. 54: 72
139. Yakubovich VA (1977-79) To the abstract theory of optimal control. I – IV (in Russian). Sib. Mat. Zh. 18:685, 19:436, 20:885, 20:1131
140. Bakke VL (1974) A maximum principle for optimal control problem with integral constraints. J. Optimiz. Th. Appl. 13:32
141. Bakke VL (1980) Boundary arcs for integral equations. J. Optimiz. Th. Appl. 19: 425
142. Bakke VL, Guinn T (1978) Optimal fields for integral equations. J. Math. Anal. Appl. 64:592
143. Balder EJ (1984) On existence problem for the optimal control of certain nonlinear integral equation of Uryson type. J. Optimiz. Th. Appl. 42:447
144. Bensoussan A (1983) Lecture on stochastic control. Lecture Notes in Math. 972
145. Bensoussan A (1987) On some parabolic equations arising from ergodic control. In: Nonlinear parabolic equations: Qualitative properties of solutions. New York, pp. 29-40
146. Bensoussan A, Lions J-L, Papanicolau G (1978) Asymptotic analysis for periodic structures. Amsterdam
147. Bernstein DS Control constraints, abnormality and improved performance by periodic control. IEEE Trans. Automat. Control AC-30
148. Besjes JC (1969) On the asymptotic methods for nonlinear differential equations. J. Méc. 8:352
149. Bismut J-M (1979) An introduction to duality in random mechanics. Lecture Notes in Control and Infrom Sci. 16:42
150. Bismut J-M (1981) Mécanique Aléatoire. Lecture Notes in Math. 866:1

151. Bittani S, Guardabassi G, Mafezzoni C, Silverman L (1978) Periodic systems: controllability and the matrix Riccati equations. SIAM J. Control and Optimiz. 16: 37
152. Bittner L (1976) Necessary optimality conditions for a model of optimal control processes. Banach Center. Publ. 1:25
153. Blankenship G, Papanicolau GC (1978) Stability and control of systems with wide-band noise disturbances. SIAM J. Math. Anal. 34
154. Blankenship G, Sachs S (1979) Singularly perturbed linear stochastic ordinary differential equations. SIAM J. Math. Anal. 10:306
155. Chaplais F (1984) Averaging et control optimale deterministe. Lecture Notes in Control and Infrom Sci. 62:502
156. Chen Y-G (1983) On the stability of perturbations of optimal control problems. J. Math. Anal. Appl. 97:46
157. Colonius F. (1986) Optimality for periodic control of functional differential systems. J. Math. Anal. Appl. 120:119
158. Dorato P, Sanchez D, Wang Y-S (1983) Approximation in policy space for periodic optimization. IEEE Trans. Automat. Control AC-28
159. Ewing GM (1982) Sufficient conditions for asymptotic optimal control. J. Optimiz. Th. Appl. 22:307
160. Gilbert EG. (1977) Optimal periodic control: a general theory of necessary conditions. SIAM J. Control and Optimiz. 15:717
161. Gommersal A, Farmer P (1984) Robotics: The international bibliography with abstracts. New York
162. Gron J. (1981) On the optimal control for nonlinear equations with impulses. Mathemat. Nachrict. 104.41
163. Guardabassi G, Locatelli A, Rinaldi S (1984) Status of periodic optimization of dynamical systems. J. Optimiz. Th. Appl. 14:1
164. Halanay A (1974) Optimal control of periodic solutions. Rev. Rouman. Mat. Pure Appl. 19
165. Haussman UG (1976) General necessary conditions for optimal control of stochastic systems. Math. Prog. Study. 6:236
166. Haussman UG (1981) On the adjoint problems for optimal control of diffusion processes. SIAM J. Control and Optimiz. 19:221
167. Haussman UG (1982) Extremal control for completely observable diffusion. Lecture Notes in Control and Infrom. Sci. 42:461
168. Haussman UG (1983) On the approximation of optimal stochastic control. J. Optimiz. Th. Appl. 40:433
169. Hestenes MR (1966) Calculus of variations and optimal control theory. New York
170. Holland CJ (1975) Gaussian open loop control problems. SIAM J. Control and Optimiz. 13
171. Itô K (1951) On stochastic differential equations. Mem. Amer. Math. Soc. 4
172. Koktovic PV (1984) Applications of singular perturbation techniques. SIAM Rev. 26:501
173. Kuo HH (1973) On the stochastic maximum principle in Banach space. J. Funct. Anal. 14:146
174. Kurtz TG (1975) Semigroup of conditional shifts and approximations of Markov processes. Am. Prob. 3:618
175. Kushner HJ (1971) Introduction to stochastic control. New York
176. Kushner HJ (1977) Probability methods for approximations in stochastic control and for elliptic equations. New York
177. Kushner HJ (1965) On the stochastic maximum principle. Fixed time of control. J. Math. Anal. Appl. 11:78
178. Kushner HJ (1978) Optimality conditions for the average cost per unit time problem.

SIAM J. Control and Optimiz. 16:78
179. Kushner HJ (1984) Approximation and weak convergence methods for random processes, with applications to stochastic systems theory. The M.I.T. Press, Cambridge, MA
180. Kushner HJ, Rungguldier WJ (1987) Filtering and control for wide bandwidth noise driven systems. IEEE Trans. Automat. Control AC-32:123
181. Kushner HJ, Rungguldier WJ (1987) Nearly optimal state feedback control for stochastic systems with wideband noise disturbances. SIAM J. Control and Optimiz. 25:298
182. Lànskà V (1983) An application of diffusion approximation to a controlled service system. In: Trans. 9th Prag. Conf. Inf. Theory, Statist. Desic. Funct.. Random Proc. Prague.
183. Maffezzoni C. (1974) Hamilton–Jacobi theory for periodic control problems. J. Optimiz. Th. Appl. 14:21
184. Matsubara M, Nishimura Y, Watanabe N, Onogi K (1981) Periodic control theory and applications (in Japanese). J. Soc. Instr. Contr. Eng. (Jap.) 20
185. Marzollo R (ed) (1972) Periodic optimization. New York
186. McShane EJ (1974) Stochastic calculus and stochastic models. New York
187. Nistri P (1983) Periodic control problems for a class of nonlinear periodic differential systems. Nonlinear Anal. Theor. Meth. Appl. 7
188. Nogueira ACP (1984) Asymptotic analysis of a certain random differential equation. Stoch. Probl. Appl. 17:229
189. Pandit G, Deo SG (1983) Differential systems involving impulses. Lecture Notes in Math. 954
190. Papanicolau GC, Kohler W (1974) Asymptotic theory of mixing stochastic ordinary differential equations. Comm. Pure Appl. Math. 27
191. Papanicolau GC, Varadhan SRS (1973) A limit theorem with strong mixing in Banach space and two applications to stochastic differential equations. Comm. Pure Appl. Math. 26
192. Roxin E, Stern LE (1982) Periodicity in optimal control and differential games. Lecture Notes in Control and Infrom. Sci. 38:234
193. Schmidt WH (1980) Notwendige Optimalitätsbedingungen für Prozesse mit zeitvariablem Integralgleichungen in Banach-Räumen. ZAMM 60:595
194. Schmidt WH (1982) Durch Integralgleichungen beschriebene optimale Prozesses mit Nebenbedingungen in Banachräumen - notwendige Optimalitätsbedingungen. ZAMM. 2:65
195. Warfield VM (1976) A stochastic maximum principle. SIAM J. Control and Optimiz. 14:803
196. Watanabe H (1983) A note on the weak convergence of solutions of certain stochastic ordinary differential equation. Lecture Notes in Math. 1021:690
197. White BS, Franklin JN (1979) A limit theorem for stochastic two-point boundary-value problem of ordinary differential equations. Comm. Pure Appl. Math. 2:253
198. Wolfersdorf L (1976) On the maximum principle for the optimal control processes described by Hammerstein integral equations with weakly singular kernels. Banach Center Publ. 1:163
199. Wonham WM (1967) Optimal stationary control of linear systems with state-dependent noise. SIAM J. Control and Optimiz. 5
200. Yoneyama T (1984) Adjoint process in stochastic optimal control problems. Int. J. Control 39:423